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A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

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CURRENT TENDENCIES OF MATHEMATICAL RESEARCH.*

BY PROFESSOR EDWARD B. VAN VLECK.

IN an address which I had the pleasure of giving at the Atlanta meeting of the American Association for the Advancement of Science, I traced the influence of Fourier's series upon the development of mathematics. This instance is only one of many in which a central thought or problem has exerted a remarkable, sometimes even a controlling influence upon the development of mathematical thought. From early Greek times I may cite the historic problems of the duplication of the cube and the squaring of the circle. After the invention of analytic geometry it became essential to determine the tangent at a point of a given curve, and this led, as you know, to the discovery of the differential calculus, one of the most wonderful instruments ever devised for scientific thought. Almost simultaneously the problem of finding the area of a curve resulted in the construction of the definite integral, a concept even wider in scope than the differential coefficient. Still earlier, and near the beginning of the mathematical renaissance, the problem afforded by the equation of the nth degree was brilliantly solved for equations of the third and fourth degrees, and the first creative impulse subsequent to Greek times was thereby given to mathematics. In the last half of the nineteenth century this same problem

* This address was read at the Quarter Centennial of the University of Chicago before a conference of the mathematical, physical, and astronomical departments.

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eventuated in the theory of groups which has become the foundation of mathematical classification.*

Examples such as these prompt us to ask: What are the central problems in the mathematical research of to-day? Each mathematical period has its own characteristics. The discovery of the calculus was followed by a notable period which I shall term the differential and gravitational epoch of mathematics, in which the consequences of the new calculus absorbed the attention, while at the same time it was developed as the handmaid of astronomy and physical science. The names of Euler on the one side, of Lagrange and Laplace on the other, serve sufficiently to specify the era and its two-fold aspects. For many reasons the last quarter of the nineteenth century might be termed the *group period,* from which we have only in part emerged. I need not tell you how necessary it became to explore geometry and analysis under the guidance of the group concept. Yet there are reservations which must be made in designating this quarter century as the group period. Likewise it is necessary to qualify any answer to the difficult question: What is the dominant problem or central thought in the research of to-day, if there be one? Nevertheless, let us be so importunate as to insist upon an answer, however foolhardy.

In the field of applied mathematics probably the "problem of three bodies" can be picked out as par excellence the present-day problem. Already in the hands of Poincaré this has given a first quickening of the fossilized methods of mathematical astronomy. In pure mathematics, to which I shall confine my attention, the number of conspicuous problems is legion, but above them all there looms, I think, in manifold aspects the problem of the *infinite set.* Analysis, geometry, and mechanics alike have been rapidly and increasingly permeated by the point set theory of Georg Cantor, in which the central core is the problem of the infinite set. More and more do the problems in these branches require, in some form or other, up-to-date knowledge of the infinite set. Consider, for example, what is called the problem of the primitive function. Given a function $F(x)$, what conditions must be imposed

* Many kinds of geometries and number fields have come to be distinguished. Their study and correlation are a feature of modern mathematical research which I have not found it convenient to discuss in this address.

upon it in order that there shall exist another or "primitive function" whose derivative is the given function? Or take the alternative question: What generalization shall be made in the notion of a derivative in order that as wide a class of functions as possible shall possess a primitive? Answers to hese questions must involve a careful study of the infinite set of discontinuities of a function.

The characteristic tendency in the thought of to-day which I have tried to grasp under the comprehensive term "Problem of the infinite set" is shown rather as a current beneath the surface than in any individual concrete problem. The average investigator must perforce seize upon any problem which his brains find at hand. There exist, however, certain fundamental principles which will aid him in finding a worthy one. The great mathematician Jacobi is said to have inculcated upon his students the dictum: Man muss immer umkehren. One must always seek a converse, turn a thought he other end to. It was by turning the elliptic integral side out that Jacobi obtained his splendid theory of elliptic nd theta functions. As other instances of the same principle ite at random the implicit function theory which under- so much modern investigation, and the perception of the ence of transcendental numbers through failure of the processes which produce our ordinary numbers.

hout dwelling further upon the fertility of Jacobi's , I wish to coin and put beside it another obvious dictum vider reach: Man muss immer generalizieren. By this mean cheap generalization to n dimensions or variables which has been already done for two or three. Not ntly, however, generalization even to n variables is a of importance and difficulty, inasmuch as the solution special case of two or three variables may possess a tive structure and character and hence be in no way al. A notable instance of such difficulty is found in generalization of the ordinary second order conditions he maximum of a definite integral in the calculus of ions, while extraordinary obstacles are encountered e extension of the theory of analytic functions to two ore variables. But whether or not extension to n dimen- be trivial, generalization to a countably infinite number dimensions becomes sublime! With a further extension an uncountably infinite number of dimensions or variables

it might seem that we would reach a sublimated or etherial stage of mathematical development, but such flippant characterization would give an altogether false impression of Volterra's theory of functions of lines.

Extension to a countably infinite number of variables commonly affords opportunity for the finest sagacity an insight. An excellent program for work could be found i extension of almost any finite theory. It is your own Professor Moore whom I have heard glowingly preach that to every finite theory there must correspond, under proper limitations, a general transcendental theory with an infinite number of variables. What is more beautiful and simple in elementary mathematics than the application of the theory of determinants to the solution of a set of n linear equations in n variables

$$(1 + a_{11})x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = C_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1n}x_1 + a_{2n}x_2 + \cdots + (a_{nn} + 1)x_n = C_n;$$

such a system as we learned to solve even before we reach quadratic equations in algebra? Now it was precisel passage from this simple system of linear equations to a sy with a countably infinite number of unknowns which to Volterra and Fredholm the basis of their theory of i equations. At the limit the system of linear equations into a linear integral equation

$$u(x) + \int_a^b u(\xi)K(x, \xi)d\xi = f(x),$$

in which everything is given except the function $u($ is to be found. The solution $(x_1, x_2, \cdots, x_n)$ of th system with the indefinite increase of n passes over i solution $u(x)$ of the integral equation, but the legit of the passage to the limit and of the solution must be fied in some manner. From the very outset the the integral equations has been found very useful in applicat mathematical physics and elsewhere. This might be ex because it is the generalization of the elemental the a set of linear equations.

The solution of integral equations is only one of r evidences of the mathematician's tightening grip upon

infinite. The infinite increase in the number of variables in our mathematical problems necessitates geometrically a theory of space of countably infinite dimensions. Already a promising beginning has been made by restricting the attention to Hilbertian space—that is, to that portion of infinitely-dimensioned space for which Σx_n^2, the sum of the squares of the coordinates of a point, is convergent. The familiar conditions for orthogonality of lines, etc., then generalize without material modification.

The theory of a space of infinitely many variables is companion to an incipient theory of functions of an infinite number of variables. I fancy a smile, inward or outward, on the faces of some of the physicists and astronomers present, to whom such a function theory may seem highly typical of the theoretic, up-in-the-air character of the mathematician and his work. Let me try to address to you some arguments ad hominem. Have you nothing of such character in physics? In my early life, when I was hesitating between mathematics and physics as a profession, I first encountered Lagrange's generalized coordinates, whose number, by the way, need not be finite. The equations for motion, energy, etc., were set up in terms of these generalized coordinates. I shall never forget how the glittering generality of the theory astonished and fascinated me. And when the equations were actually applied to sundry objects—as, for instance, to the gyroscope —it seemed to me then, as it almost does still, about the finest example of getting something intellectually out of nothing that I had ever come across. The very generality of the coordinates gives to them adaptability to the varying complications of physical science. Now most phenomena of nature depend in their final analysis upon an infinite number of variables. How then can it be thought that we can master them unless the mathematician prepares the way by developing a theory of functions of infinitely many variables? The potential or force due to an electric current in a wire is a function of the shape of the wire and depends therefore upon the uncountably infinite number of its points. Indeed, it is not the pure mathematician who has been so hardy and rash as to attempt the beginning of a theory of functions of lines. No, the bold pioneer in this new field is no other than a mathematical physicist or, as we may claim with equa truth, a physical mathematician—Volterra, a superb interpreter of current tendencies.

Time fails me to speak of the closely related and fundamental "functional calculus" initiated by Fréchet, certain aspects of which are being developed here at Chicago by Moore and his pupils.

An effect, and likewise a cause, of the generalized aims in mathematics, of which I have thus far spoken, has been the creation of refined mathematical tools. Under the term "mathematical tool" I mean to include concepts and algorisms which are usable for the demonstration of mathematical truth. The importance of concepts broadly comprehensive in their reach is shown alike by the history of geometry and of analysis. The open plane of Euclid yields a general geometry only after the introduction of ideal elements which we call the point, line, etc., at infinity. The creation thereby of a closed and perfect projective geometry was one of the milestones of mathematical progress in the century just past. Poncelet's comprehensive principle of geometric continuity, justified at first rather by necessity than by adequate logic, was another indispensable geometric creation. In the notion of a group we have a concept which reaches pervasively through the realms of geometry and analysis. Still another example of the fecundity of the subtler concepts forged by the progress of science is to be found in Hill's "periodic orbit," introduced in the theory of attraction as a norm from which to reckon the deviation of the orbit of the heavenly body.

My remaining illustrations will be taken chiefly from analysis, partly because it is the province with which I am most familiar, but also because the same general tendency which was exhibited conspicuously in the magnificent enlargement of geometry in the half century from 1820 to 1870 is to-day manifested in the broadening of analysis. Consider the familiar notion of an integral, apparently moulded to a finality in the hands of Riemann. This admits a finite number of discontinuities but an infinite number only under certain narrow restrictions. A totally discontinuous function—for example, one equal to zero in the rational points which are everywhere dense in the interval of integration, and equal to 1 in the irrational points which are likewise everywhere dense—is not integrable à la Riemann. The restriction became a very hampering one when mathematicians began to realize that the analytic world in which theorems are deducible does not consist merely of highly civilized and

continuous functions. In 1902 Lebesgue with great penetration framed a new integral which is identical with the integral of Riemann when the latter is applicable but is immensely more comprehensive. It will, for instance, include the totally discontinuous function above mentioned. This new integral of Lebesgue is proving itself a wonderful tool. I might compare it to a modern Krupp gun, so easily does it penetrate barriers which before were impregnable.

Some of the newer mathematical concepts resemble the modern telescopes, disclosing new worlds of theorems and ideas bound together in a common system. Other concepts because of their high dispersive power act more like the optical grating, making new distinctions visible. This is particularly true of the modern theory of point sets which gives a grip on weird sets of points almost inconceivable in their irregularity. Take for consideration the two sets of points of which I spoke a few moments ago, the set of rational points everywhere dense and the set of irrational points likewise everywhere dense in a given interval, confusedly mixed together like grains of pepper and salt. Which predominates? Should the mixture be called white or black? Would it seem possible to say? Yet the theory of point sets tells us that the black—i. e., the irrational points—predominate; that the rational points are the exception and have a measure 0, while the irrational points have a measure equal to the length of the interval. The rational points are weighed in the balance by Borel and Lebesgue and are found altogether lacking, at least in measure. They afford an example of what may be termed a null set, that is, a set of measure 0. It is natural to expect that such a set of points would be without influence on many results—for example, on the value of a Lebesgue integral. This turns out to be the case. By allowing for a set of exceptional points of measure 0 many theorems can now be perceived and demonstrated which before were unthinkable.

Along with the widened aims and concepts of mathematics have come fresh methods and distinctive styles of thought. Look first at the older methods. The work of the grand old mathematicians of the seventeenth and eighteenth centuries was based on definite algorisms, such as the hypergeometric series of Gauss and the Taylor power series. Euler, in particular, was astonishingly prolific in algorisms. When the

function theory sprang into existence it became even more indispensable to make elaborate studies of special algorisms and functions. A host of particular functions absorbed the attention of mathematicians—elliptic functions, automorphic functions, Bessel's functions, other functions defined by differential equations, and so on. The function theory of Weierstrass was based on a definite algorism, the Taylor power series. In recent times other processes and functions have received attention, for example, algebraic continued fractions and divergent series, factorial series, Dirichlet series, and so on. Definite algorismic work must, indeed, always remain of primary importance, not only on account of its use in calculation but also because the progress of our science continually introduces new functions for study. At the same time older processes need to be reexamined under modern lights,—for instance, the difference equation which has been considered by Nörlund, Birkhoff, and Carmichael in the light of the analytic function theory.

In sharp contrast with this algorismic work so characteristic of earlier time is a distinctively modern turn of thought which perhaps first found a clear exponent in Riemann. He presents a strange antithesis to his contemporary countryman, Weierstrass. Riemann bases the function theory upon a property rather than upon an algorism—to wit, the possession of a differential coefficient by the function in the complex plane. Thus at a stroke it is freed from dependence upon a particular process like the power series of Taylor. His celebrated memoir upon the P-function is a characteristic development of a whole Schar (family) of functions from their mutual relations.

A first aspect of the peculiar thought-mode of which Riemann's is a type is definition and development of a function from its properties. One illustration is found in the definition of a function by building a given region point for point and conformally upon another given region. In severe cases the regions may even contain an infinite number of leaves properly bound together. This illustration is significant also of a certain modern tendency to build mathematics upon correspondence and order. The boundary value problems of mathematical physics afford another example of definition of a function through its properties. As an effect of this tendency we may expect ultimately a great develop-

ment of the incipient theory of functional equations. In this theory a single equation or system of equations expressing some property is taken as the definition of a class of functions whose characteristics, particular as well as collective, are to be developed as an outcome of the equations.

A second phase of the same thought tendency is to be found in the existence theorems so characteristic of modern analysis. Existence proofs are, indeed, a preliminary and indispensable element, if functions are to be defined through their properties. Their rôle has been discussed so often that I need not enlarge upon it.

A third phase of the same tendency is the modern examination of the postulates of geometry, arithmetic, algebra, and even mechanics. This postulational work seizes the material at the root before development sets in. It has to face the obvious objection that it is an exercise in sterile logic rather than creative mathematics. A swirling brain like Poincaré's will never stop in its constructive work for abstract postulational considerations, but will leap gaps like an electric spark. To this objection the rejoinder may be made that critical revision is not so far from creative thought as is generally supposed; that postulational theories are to-day much more fertile than in the past, that such postulational developments as Hilbert's in his famous Foundations of Geometry and his foundations for a geometric group theory are, in fact, creative work of highest order; that Lebesgue formulated his concept of a generalized integral in the light of postulational requirements; and so on.

In this connection the primal thought which lies at the bottom of Moore's General Analysis deserves special consideration. When, he argues, a number of similar results in different fields are compared, the belief is forced irresistibly upon one that the essential oneness of the results must be due to identity of the hypotheses, this identity being partially or altogether veiled from sight by the admixture of foreign ingredients, as in chemical compounds. To apply a technical term, we have before us theories to be rendered simply isomorphic. The problem before the mathematician is accordingly to extract the common basis and exhibit the results as consequences of a common set of hypotheses applicable to varying conditions and objects. Thereby not only will like theories be coordinated, but the common underlying prin-

ciples laid bare will be ready for use elsewhere. Doubtless the mathematician will continue to be guided for the most part by analogy in the future as in the past, but I doubt not that the isolation of the common elements in like theories will be stimulating and productive, even as the postulational study of the foundation of geometry has led to the discovery of new geometries. The general analysis has furthermore the advantage over ordinary postulational investigations of not tapping the root but the sap.

The change in method of which I have spoken centers largely in the prescription of properties and postulates rather than of algorisms. Entirely different from this is the increasing tendency to recast a problem so as to make it fertile. In the words which I have heard Hadamard employ, a problem must be "bien posé," fittingly formulated in order to give rich result. The inner nature of the problem needs to be comprehended. It carries in itself its own inherent limitations on its solution. To illustrate by drawing upon the calculus of variations, the problem of minimizing the integral

$$\int F(x, y, y', y'', \cdots, y^{(n)})dx$$

by a function $f(x)$ is badly formulated and does not admit of precise answer unless we specify the class of functions from which the solution is to be taken. If we take, for example, the solution from functions of class $C^{(n)}$ in Bolza's terminology —that is, functions which are continuous together with their derivatives down to the nth inclusive—the problem then becomes well formulated and can be tackled successfully. In this particular case we impose limitations in order to well-formulate the problem. But it is distinctly modern to make a problem solvable by widening it so as to make it more instead of less comprehensive. Thus a problem not possible of general solution if only functions integrable in the ordinary Riemannian sense are considered, may become so if Lebesgue integrals are admitted. In fact, the solution may depend upon the kind of tool employed, and a careful study of the adaptation of the tool to the problem and of the problem to the tool should therefore be made. Thus if the problem should involve convergence, its character and formulation may change according as the tool involved is ordinary convergence, uniform convergence, quasi-uniform

convergence, convergence "en moyen," or essentially uniform convergence.

In conclusion, partly by way of illustration and partly by way of summary, let me present a theorem which embodies most of the tendencies of which I have spoken. This is the so-called Fischer-Riesz theorem, remarkable for its elegant character. You all know the representation of a function $f(x)$ by a Fourier series,

$$\frac{a_0}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

in which the coefficients are obtained from the function by the equations

$$a_n = \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad b_n = \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Now as Jacobi said, Man muss immer umkehren. Instead of representing a known function by a Fourier series, let us seek conversely to represent a given Fourier series by a function. The given series may be either convergent or divergent. We have now before us one of the great problems of analysis.

But before considering the problem let us obey the injunction: Man muss immer generalizieren. The Fourier series is only one of a class of series

$$\text{(I)} \qquad a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \cdots,$$

in which the functions have with respect to a given interval (a, b) the so-called orthogonality property,

$$\int_a^b \phi_i(x)\phi_j(x)dx = 0 \qquad (i \neq j).$$

If, further, the functions have been normalized, as was the case above in the Fourier series, we have also

$$\int_a^b \phi_i^2(x)dx = 1.$$

Such a series is called a *generalized Fourier's series*, of which many special cases are well known, as, for instance, a series in terms of Legendre polynomials.

The problem now before us is to represent such a series, whether convergent or divergent, by a function and to derive

the function from the series. We shall agree to say that a function $f(x)$ represents the series if 1) it is connected with the series by the equations

$$a_n = \int_a^b \phi_n(x) f(x) dx$$

and thus generates the series in the customary way; and if 2) the function agrees in value with the series at all points of the interval (a, b) for which the series is convergent. The questions now confront us: When will such a function exist? How can it be obtained from the series? To what extent is it unique? In this shape the problem would be almost or quite insoluble. The problem is not yet "bien posé." For good formulation we must change the tool and use Lebesgue integrals throughout, thus extending the class of functions admissible for the representation of the series. Furthermore, in deriving the function from the series we will not require that the function and the series shall agree at all points of convergence but will allow exceptions at an infinite number of points forming a set of measure 0.

It is now possible to state the theorem which Fischer and Riesz obtained independently by different methods. The necessary and sufficient condition that there shall exist a function representing the series (and having an integrable square $f^2(x)$) is that the sum of the square of the coefficients, Σa_n^2, shall be convergent; in other words, the point $(a_1, a_2, \cdots)$ must lie within Hilbertian space of infinitely many dimensions. The function is completely determined save at an arbitrary set of points of measure zero, where we can give the function any value we choose. It can be obtained from the series by a proper grouping of the terms without change of order, which will render the series convergent except possibly at a set of points of measure 0. The study of the convergence of the series has brought to light two new modes of convergence, one of which, Weyl's essentially uniform convergence, seems to me very important. The other I shall stop to explain briefly because it is an admirable example of generalized concept and because I think that it may interest those of you who are acquainted with the theory of least squares. If $S_n(x)$ denotes the sum of n terms of our generalized Fourier series (I) and if $f(x)$ is, as before, the function represented by the series, the square of the error at any

point x of the interval is $[S_n(x) - f(x)]^2$, and the sum of the squares of the errors taken over the interval is the integral $\int_a^b [S_n(x) - f(x)]^2 dx$. As n increases indefinitely, this error integral approaches 0, so that the series converges "en moyen" (on the average) although there may be no true convergence in the ordinary sense of the term. Conversely, if

$$\int_a^b [S_{n+m}(x) - S_n(x)]^2 dx$$

can be made as small as we please by taking n sufficiently great, then there exists a function $f(x)$ toward which the series converges en moyen.

But enough of this. I trust that you see that the Fischer-Riesz theorem affords an elegant example of modern mathematical research—characteristic in its generalized sweep, in its creation and use of refined tools and concepts, in its recasting of the problem so as to make it fertile, and lastly in its attainment of success through point set considerations, involving in this case the recognition of a null set of exceptional points. Not a little of the work of analysts for some time to come must center around the numberless new notions until their "Tragweite"—their carrying power and reach—is determined. To a great extent they have modified the character of current investigation. There is regretably less of the action and interaction of mathematics and mother nature. There is also insufficient study of approximation and astronomical mathematics—fields which are fortunately represented here by Moulton and MacMillan and which serve to counteract or supplement a somewhat widespread abstruse inclination in our country. But the newer thought does make tremendous advance when questions of theory rather than of calculation are involved, and the combination of sweeping generalization with rigor is astonishing. There remains yet to be accomplished the complete interweaving and correlating of the old and the new. We cannot see far into the future, but the well of mathematics will not run dry as long as there remains such springs as the problem of three or n bodies, and the tantalizing, old, yet unsolved problem of the shape of curves of the nth degree.

UNIVERSITY OF WISCONSIN,
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GRAPHICAL METHOD OF FINDING THE POSSIBLE SETS OF INDEPENDENT GENERATORS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 22, 1916.)

If the order of an abelian group G is not a power of a prime number, a set of independent generators of G can be obtained by taking an arbitrary set of independent generators of each of the Sylow subgroups of G and regarding the totality of operators in the sets thus obtained as a single set. This is true irrespective of which of the two common definitions of a set of independent generators may be adopted. According to one of these two definitions any set of l operators of G which generate G and are such that no $l-1$ of them generate G is said to be a set of independent generators of G. This is sometimes called the general definition of a set of independent generators, and every possible group has at least one such set of independent generators. According to the restricted definition of a set of l independent generators, the additional condition is imposed that the group generated by every $l-1$ of these operators has only identity in common with the group generated by the remaining one.

It is evident that not every possible group has a set of independent generators according to the restricted definition. For instance, the quaternion group does not have any such set of independent generators. It is well known that every possible abelian group has at least one set of independent generators according to each of these two definitions, and that the number of the operators in such a set is the same under both definitions whenever the group is of prime-power order and has at least one set of independent generators satisfying both definitions. The latter statement remains true even when the group of prime-power order is non-abelian.

Since the determination of a set of independent generators of an abelian group whose order is not a power of some prime number may be made to depend upon the determination of such a set of independent generators for each of its Sylow subgroups, irrespective of which of the two given definitions

of a set of independent generators is adopted, we shall confine ourselves, in what follows, to the consideration of the possible sets of independent generators of an abelian group G whose order is p^m, p being a prime number. For brevity it will always be assumed, in what follows, that the independent generators under consideration satisfy the restricted definition unless the contrary is explicitly stated.

Let H be the characteristic subgroup of G composed of the operators obtained by raising each operator of G to its pth power. The quotient group G/H is of type $(1, 1, 1, \cdots)$, and if the order of this quotient group is p^k, G has exactly k independent generators. Moreover, whenever G has exactly k independent generators the order of this quotient group is p^k. None of the operators of H can be used as an independent generator of G, irrespective of which of the two definitions of independent generators is adopted. According to the general definition of independent generators, H is composed of all the operators of G which cannot appear in some one of the possible sets of independent generators of G; that is, H is the ϕ-subgroup of G.

A necessary and sufficient condition that every operator of G which does not occur in H appears in at least one of the possible sets of independent generators of G is that no two invariants of G have a quotient which exceeds p, and a necessary and sufficient condition that all the operators of G which are not contained in H are conjugate under the group of isomorphisms of G is that all the invariants of G are equal to each other. When the latter condition is satisfied a set of independent generators of G can be chosen in

$$p^{mk-k^2}(p^k - 1)(p^k - p)(p^k - p^2) \cdots (p^k - p^{k-1})$$

different ways. The first of the factors in this product represents the number of ways in which these independent generators can be selected after the co-sets from which they are to be taken have been determined, while the product of the other factors represents the number of ways in which these co-sets can be chosen.

Since the order of the group of isomorphisms of any abelian group is equal to the number of different ways in which a set of independent generators of the group can be selected, it results that the above formula represents the order of the group of isomorphisms of G whenever all the k invariants

of G are equal to each other and only then. The same formula clearly represents the number of ways in which a set of general independent generators of G can be chosen irrespective of whether the k invariants of G are equal to each other or do not have this property.

Suppose that a set of independent generators of G contains k_1 operators of order p^{α_1}, k_2 of order p^{α_2}, $\cdots$, k_λ of order p^{α_λ}, $\alpha_1 < \alpha_2 < \cdots < \alpha_\lambda$. Then $k_1 + k_2 + \cdots + k_\lambda = k$, and G contains exactly $p^{k_1} - 1$ co-sets with respect to H which involve operators of order p^{α_1} but no operators of lower order. These co-sets together with H constitute a subgroup H_1 of G which includes all the operators of G whose orders do not exceed p^{α_1}. The co-sets which involve operators of order p^{α_2}, but no operators of lower order, together with H_1 constitute a subgroup H_2 of G which includes all the operators of G whose orders do not exceed p^{α_2}, etc.

A necessary and sufficient condition that any given operator of G appears in at least one set of independent generators is that it belongs to a co-set of G with respect to H and that it is one of the operators of lowest order in this co-set. If such an operator is of order p^α, the co-set in which it occurs has as many operators of order p^α as there are operators in H whose orders divide p^α, $\alpha_1 \leqq \alpha \leqq \alpha_\lambda$. Hence the following diagram

<table>
<tr><td colspan="2" rowspan="2">H</td><td colspan="4">Co-sets of G as regards H whose smallest operators are of order</td></tr>
<tr><td>p^{α_1}</td><td>p^{α_2}</td><td>. . . . . .</td><td>p^{α_λ}</td></tr>
<tr><td rowspan="4">Operators of H whose orders divide</td><td>p^{α_1}</td><td>Possible independent generators of order p^{α_1}</td><td rowspan="2">Possible independent generators of order p^{α_2}</td><td rowspan="4">. . . . . .</td><td rowspan="4">Possible independent generators of order p^{α_λ}</td></tr>
<tr><td>p^{α_2}</td><td></td></tr>
<tr><td>. . . . . .</td><td></td><td></td></tr>
<tr><td>$p^{\alpha_\lambda - 1}$</td><td></td><td></td><td></td></tr>
</table>

may serve to exhibit the operators from which a set of independent generators must be selected:

This diagram exhibits not only the operators from which the independent generators of different orders must be selected and the operators which cannot be used as such generators, but also the number of ways in which the independent generators of any possible order can be chosen. For instance, if G contains independent generators of order p^{a_γ} and if $p^{a'_\gamma}$ is the order of the subgroup of H which is composed of all its operators whose orders divide p^{a_γ}, then the k_γ generators of order p^{a_γ} can be chosen in a number of ways represented by the following product:

$$p^{(a'_\gamma+k_1+k_2+\cdots+k_{\gamma-1})k_\gamma}(p^{k_\gamma}-1)(p^{k_\gamma}-p)\cdots(p^{k_\gamma}-p^{k_\gamma-1}),$$

where $k_{\gamma-1}=0$ when $\gamma=1$. The order of the group of isomorphisms of G is therefore equal to

$$\prod_{\gamma=1}^{\gamma=\lambda} p^{(a'_\gamma+k_1+k_2+\cdots+k_{\gamma-1})k_\gamma}(p^{k_\gamma}-1)(p^{k_\gamma}-p)\cdots(p^{k_\gamma}-p^{k_\gamma-1}).$$

Instead of considering the co-sets of G with respect to H we may consider the co-sets of H_1 with respect to H, then those of H_2 with respect to H_1, etc. Finally, those of G with respect to $H_{\lambda-1}$ are considered for the purpose of finding the possible independent generators of highest order. According to this method, the determination of the possible sets of independent generators of G is simply a repetition of the process employed to determine the independent generators of G which are of order p^{a_1} and are contained in H_1. This method of determining the possible sets of independent generators of G has contact with the method described by the writer in the *Tôhoku Mathematical Journal*, volume 2, 1912, page 137. Its main advantage is due to the fact that it exhibits clearly the operators which can appear in some possible set of independent generators.

UNIVERSITY OF ILLINOIS.

ON THE DEVELOPMENTS IN BESSEL'S FUNCTIONS.

BY PROFESSOR CHARLES N. MOORE.

(Read before the American Mathematical Society, January 1, 1916.)

In studying the developments in Bessel's functions it is found that the behavior of the series at the origin and in a neighborhood including the origin is, in certain cases, difficult to determine. For example most of the proofs of the convergence of these developments leave unsettled the question of the convergence at the origin of the developments in Bessel's functions of order zero. Also, most of the discussions of the uniform convergence of these developments leave unsettled the question of the uniform convergence of the developments in a neighborhood including the origin. Yet for many of the applications to mathematical physics it is essential to settle both of these points in order to be sure that we have a solution of the physical problem we are discussing.

It was a consideration of these facts that led the writer several years ago to devise a method of establishing the convergence of the developments in Bessel's functions that would settle both of these questions under conditions sufficiently wide for the applications.* The method referred to, however, serves only to establish the convergence or uniform convergence of the developments, and it is necessary to determine the value to which they converge by means of other considerations. In the paper just referred to this was done by using the results of previous writers in connection with the value of the developments. As these results are obtained by means of long and complicated discussions, it seems highly desirable to have some fairly simple method of determining the value of the development when we know it converges. Such a method is given in the present paper, and consequently this paper combined with the previous one gives the first *complete* discussion of the convergence and value of the developments in Bessel's functions under conditions that are usually satisfied in the applications.

* Cf. *Transactions*, vol. 12 (1911), p. 181.

The method used here was first suggested by Sturm and Liouville in connection with the Sturm-Liouville expansions, and the application of it to these expansions was sketched out by them.* Some forty years later Heine attempted to apply this method to the developments in Bessel's functions.† His treatment of this case is inadequate, however, for by omitting practically all the details he avoids all the difficulties. Moreover, it is apparent that he did not himself examine the details of the proof with sufficient care, for he fails to note the presence of an extra term in the development in certain exceptional cases, a fact first observed by Dini.‡

The cases just mentioned are not essentially exceptional but only appear so on account of the notation ordinarily employed. As pointed out before,§ they can be included in the general treatment by means of adopting the following notation. We set

$$(1) \qquad F_\nu(\lambda, x) \equiv \left(\frac{x}{2}\right)^\nu \sum_{i=0}^{\infty} \frac{(-1)^i (\lambda x)^{2i}}{2^{2i} i! \, \Gamma(\nu + i + 1)} \equiv \frac{1}{\lambda^\nu} J_\nu(\lambda x)$$

and consider the series

$$(2) \qquad \sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 x f(x) F_\nu(\lambda_n, x) dx}{\int_0^1 x [F_\nu(\lambda_n, x)]^2 dx} = \sum_{n=1}^{\infty} A_n F_\nu(\lambda_n, x),$$

where the λ's are the successive roots, positive or zero of the equation

$$(3) \qquad \omega(\lambda) = \left[l \frac{\partial}{\partial x} F_\nu(\lambda, x) + h F_\nu(\lambda, x) \right]_{x=1} = 0,$$

arranged in increasing order of magnitude.

The object of the present paper is to show that whenever the series (2) converges or is summable at all points of the interval $0 \leqq x \leqq 1$ at which $f(x)$ is continuous, then under certain conditions to be stated subsequently the value of the series at each of these points will be $f(x)$. We will begin by establishing some lemmas.

* Cf. *Liouville's Journal*, vol. 2 (1837), p. 220.
† Cf. *Crelle's Journal*, vol. 89 (1880), p. 35.
‡ Dini, Serie di Fourier, Pisa, 1880.
§ Cf. *Transactions*, vol. 10 (1909), p. 420, and vol. 12 (1911), p. 182.

LEMMA 1. *The integrals*

$$(4)\qquad \begin{aligned} &\frac{1}{2\pi i}\int_{R_n}\frac{zF_\nu(z, x)}{(z-\alpha)\omega(z)}dz, \\ &\frac{1}{2\pi i}\int_{R_n'}\frac{F_\nu(z, x) - x^\nu F_\nu(z, 1)}{(z-\alpha)z^2F_\nu(z, 1)}dz \quad (\alpha \text{ a real constant}), \end{aligned}$$

where $\omega(z)$ *is defined by* (3), l *not being zero, and the path of integration is a rectangle in the complex plane with vertices at the points* $(k, \sqrt{k})$, $(-k, \sqrt{k})$, $(-k, -\sqrt{k})$ *and* $(k, -\sqrt{k})$, k *being chosen equal to*

$$(5)\quad n\pi + \frac{2\nu+3}{4}\pi \quad \text{or} \quad n\pi + \frac{2\nu+1}{4}\pi \quad (n \text{ a positive integer}),$$

according as we are considering the first integral or the second integral, will approach zero as n *becomes infinite, provided* x *lies in the interval* $0 < x < 1$.

We shall give the proof in detail for the first only of the integrals in (4), the treatment of the second integral being analogous.

We have for $J_\nu(z)$ the following asymptotic expansion*

$$(6)\quad J_\nu(z) = \frac{1}{\sqrt{2\pi z}}[e^{i(z-\gamma)}(1+\theta_1(z, \nu)) + e^{-i(z-\gamma)}(1+\theta_2(z, \nu))],$$

where $\gamma = \frac{1}{4}(2\nu+1)\pi$ and

$$(7)\qquad |\theta_1(z, \nu)| < \frac{K_1}{|z|}, \quad |\theta_2(z, \nu)| < \frac{K_2}{|z|} \quad \begin{pmatrix} |z| > c > 0 \\ R(z) > 0 \end{pmatrix},$$

K_1 and K_2 being positive constants for a fixed value of ν.

Making use of (1) and the well-known relation

$$(8)\qquad J_\nu'(z) = -J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z),$$

we obtain

* Cf. H. Weber, "Zur Theorie der Bessel'schen Functionen," *Math. Annalen*, vol. 37 (1890), p. 404. Equation (6) above results from a combination of equations (19), (30) and (33) of the paper referred to, and is established there for all values of $\nu > -\frac{1}{2}$, and for all values of z whose real part is positive. It can be at once extended to values of z along the axis of imaginaries, the origin being excepted, on account of the continuity of all the functions involved, and for values of z to the left of the axis of imaginaries we can get a corresponding expression by means of the relationship $J_\nu(-z) = (-1)^\nu J_\nu(z)$.

$$\omega(z) = \frac{1}{z^\nu}[(l\nu + h)J_\nu(z) - lzJ_{\nu+1}(z)]. \tag{9}$$

For the sake of brevity we set

$$\theta_1 = \theta_1(\pm z, \nu), \quad \theta_1' = \theta_1(\pm zx, \nu), \quad \theta_2 = \theta_2(\pm z, \nu),$$

$$\theta_2' = \theta_2(\pm zx, \nu), \quad \theta_3 = \theta_1(\pm z, \nu + 1), \quad \theta_4 = \theta_2(\pm z, \nu + 1).$$

Then if we further set $z = p + qi$ and make use of (1), (6), the extension of (6) indicated in the previous footnote and (9), we see that the integrand of the first integral in (4) may be written in the form

$$\frac{z[e^{i(\pm px-\gamma)}e^{\mp qx}(1 + \theta_1') + e^{-i(\pm px-\gamma)}e^{\pm qx}(1 + \theta_2')]}{\sqrt{x}(z - \alpha)[\mp lz\{e^{i[\pm p-\gamma-(\pi/2)]}e^{\mp q}(1 + \theta_3) + e^{-i[\pm p-\gamma-(\pi/2)]}e^{\pm q}(1 + \theta_4)\} + (l\nu + h) \times \{e^{i(\pm p-\gamma)}e^{\mp q}(1 + \theta_1) + e^{-i(\pm p-\gamma)}e^{\pm q}(1 + \theta_2)\}]}. \tag{10}$$

For points on the sides of the rectangle R_n that are parallel to the y-axis, i. e., for values of z of the form $\pm k + iq$, where k has the first of the values (5), (10) reduces to

$$(-1)^n \frac{e^{\mp qx}e^{i(kx-\gamma)}(1 + \theta_1') + e^{\pm qx}e^{-i(kx-\gamma)}(1 + \theta_2')}{\mp l\sqrt{x}\{\pm((n+\frac{1}{2})\pi + \gamma) - \alpha + iq\}(e^q + e^{-q})(1 + \theta_5)},$$

where for large values of $|z|$, θ_5 is an infinitesimal of the same order as the other θ's. It is readily seen from this latter form that if n is chosen large enough to make θ_1', θ_2' and θ_5 each less in absolute value than $\frac{1}{2}$, the integrand in (4) will for the values of z under consideration be less in absolute value than

$$\frac{6}{l\sqrt{x}\{(n + \frac{1}{2})\pi + \gamma - \alpha\}}.$$

Hence the contributions to the first integral in (4) from the integrations along the sides of the rectangle parallel to the y-axis, will be less in absolute value than

$$\frac{1}{2\pi} \cdot \frac{24\sqrt{(n + \frac{1}{2})\pi + \gamma}}{l\sqrt{x}\{(n + \frac{1}{2})\pi + \gamma - \alpha\}}, \tag{11}$$

and therefore will approach zero as n becomes infinite.

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For points on the sides of the rectangle R_n that are parallel to the x-axis, i. e., for values of z of the form $p \pm i\sqrt{k}$, k having the same value as above, (10) reduces to*

$$\frac{e^{i(\pm px-\gamma)}e^{\mp(\pm)\sqrt{k}x}(1+\theta_1')+e^{-i(\pm px-\gamma)}e^{\pm(\pm)\sqrt{k}x}(1+\theta_2')}{\mp l\sqrt{x}\left(p-\alpha+(\pm)\,i\sqrt{(n+\frac{1}{2})\pi+\gamma}\right)e^{\sqrt{k}}e^{\mp(\pm)x[\pm p-\gamma-(\pi/2)]}(1+\theta_6)},$$

where for large values of $|z|$, θ_6 is an infinitesimal of the same order as the other θ's. It is readily seen from this form that if n is chosen large enough to make θ_1, θ_2, and θ_6 each less in absolute value than $\frac{1}{2}$, the integrand of the first integral in (4) will for the values of z under consideration be less in absolute value than†

$$\frac{6}{l\sqrt{x((n+\frac{1}{2})\pi+\gamma)}e^{(1-x)\sqrt{(n+\frac{1}{2})\pi+\gamma}}}.$$

Hence the contributions to the first integral in (4) from the integrations along the sides of the rectangle parallel to the x-axis, will be less in absolute value than

$$\frac{1}{2\pi}\cdot\frac{24\{(n+\frac{1}{2})\pi+\gamma\}}{l\sqrt{x((n+\frac{1}{2})\pi+\gamma)}e^{(1-x)\sqrt{(n+\frac{1}{2})\pi+\gamma}}}, \tag{12}$$

and therefore will approach zero as n becomes infinite.

Combining the previous results it is seen that the first integral in (4) is for large enough values of n less in absolute value than the sum of (11) and (12). Hence it approaches zero as n becomes infinite, and thus our lemma is proved for this case. As stated before, the proof for the other case is analogous.

LEMMA 2. *The function $J_\nu(\alpha x)$, where α is a real constant $\neq 0$, can be expanded into a series of the form*

$$\sum_{n=1}^{\infty} B_n F_\nu(\lambda_n, x), \tag{13}$$

* The ambiguous signs enclosed in parentheses form an independent set from those not so enclosed.

† For the corresponding values of z the integrand of the second integral in (4) can be shown to be less in absolute value than

$$\frac{12}{\sqrt{x(n\pi+\gamma)^3}}.$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ *are the roots, positive or zero, of equation* (3) *for the case* $l \neq 0$, *arranged in increasing order of magnitude, the B's are constants, and the series converges uniformly in the interval* $0 \leqq x \leqq 1$.

Let us set

$$\frac{\omega(z)}{z} = u(z) = \frac{v(z)}{z^{\nu+1}}. \tag{14}$$

Then if we make use of Cauchy's theorem of residues, we obtain for the first integral in (4)

$$\frac{1}{2\pi i}\int_{R_n} \frac{F_\nu(z, x)}{(z-\alpha)u(z)}dz = \frac{F_\nu(\alpha, x)}{u(\alpha)} + \sum_\lambda \frac{F_\nu(\lambda, x)}{(\lambda-\alpha)u'(\lambda)}, \tag{15}$$

where the summation on the right-hand side is extended over all values of λ that are roots of the first order of $u(\lambda)$ and lie within the rectangle R_n.

Since for every positive root of (3) there is a corresponding negative root, equal in numerical value, there will be two terms on the right-hand side of (15) corresponding to each positive root of (3). We shall now combine these two terms into one of the same type as the general term of (13).

Since from (14) any positive root of $\omega(z)$ is also a positive root of $u(z)$ and $v(z)$, we have for any positive root λ of (3)

$$u'(\lambda) = \frac{v'(\lambda)}{\lambda^{\nu+1}} - \frac{(\nu+1)v(\lambda)}{\lambda^{\nu+2}} = \frac{v'(\lambda)}{\lambda^{\nu+1}}. \tag{16}$$

But from (14) and (9)

$$v'(\lambda) = (l\nu + h)J_\nu'(\lambda) - lJ_{\nu+1}(\lambda) - l\lambda J'_{\nu+1}(\lambda),$$

which by means of (8) and another well-known formula,

$$J'_{\nu+1}(z) = J_\nu(z) - \frac{\nu+1}{z}J_{\nu+1}(z),$$

may be put in the form

$$v'(\lambda) = -hJ_{\nu+1}(\lambda) + \frac{l(\nu^2-\lambda^2)+h\nu}{\lambda}J_\nu(\lambda). \tag{17}$$

Moreover, since λ is a positive root of (3) and the l of (3) has been assumed to be not zero, it follows from (9) that

$$J_{\nu+1}(\lambda) = \frac{l\nu+h}{l\lambda}J_\nu(\lambda). \tag{18}$$

Combining (16), (17) and (18), we obtain

$$u'(\lambda) = \frac{l^2(\nu^2 - \lambda^2) - h^2}{l\lambda^{\nu+2}} J_\nu(\lambda). \tag{19}$$

It follows from (1) and (19) that

$$F_\nu(-\lambda, x) = F_\nu(\lambda, x), \qquad u'(-\lambda) = u'(\lambda).$$

Hence the two terms on the right-hand side of (15) corresponding to λ and $-\lambda$ combine into the form

$$\frac{2\alpha l\lambda^2}{(\lambda^2 - \alpha^2)\{l^2(\nu^2 - \lambda^2) - h^2\}F_\nu(\lambda, 1)} F_\nu(\lambda, x). \tag{20}$$

We know from Lemma 1 that for values of x in the interval $0 < x < 1$ the left-hand side of equation (15) approaches zero as n becomes infinite. Hence the right-hand side does also, and if we take limits we obtain for $J_\nu(\alpha x)$ a series of the form (13),* where

$$B_n = \frac{2\alpha^{\nu+1}u(\alpha)l\lambda^2}{(\lambda^2 - \alpha^2)[l^2(\lambda^2 - \nu^2) + h^2]F_\nu(\lambda, 1)}, \tag{21}$$

and we know that the series converges to $J_\nu(\alpha x)$ in the interval $0 < x < 1$. It follows at once from the form of B_n and well-known properties of the roots of (3) that the series converges *uniformly* throughout the interval $0 \leqq x \leqq 1$, and hence it represents $J_\nu(\alpha x)$ throughout that interval. Our lemma is therefore proved.

LEMMA 3. *The function $J_\nu(\alpha x) - x^\nu J_\nu(\alpha)$, where α is a real constant $\neq 0$, can be expanded into a series of the form* (13) *where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the successive positive roots of equation* (3) *for the case $l = 0$, arranged in increasing order of magnitude, the B's are constants, and the series converges uniformly in the interval $0 \leqq x \leqq 1$.*

If we treat the second integral in (4) in the same way as we treated the first integral in the proof of Lemma 2, we obtain for the function $J_\nu(\alpha x) - x^\nu J_\nu(\alpha)$ a series of the form (13) where

* It is apparent from (9) and (14) that $u(z)$ will have a zero root of the first order when and only when (3) has a zero root. As would be expected, the coefficient of the general term of the series (2) for the case where $f(x) = J_\nu(\alpha x)$ may be reduced to the form (21).

$$B_n = \frac{2\alpha^{\nu+2}F_\nu(\alpha, 1)}{\lambda^2(\lambda^2 - \alpha^2)F_{\nu+1}(\lambda, 1)}, \tag{22}$$

and we know that the series converges to the function in question throughout the interval $0 < x < 1$. It follows directly from the form of B_n and well-known properties of the roots of $J_\nu(\lambda)$ that the series converges uniformly throughout the interval $0 \leqq x \leqq 1$. Hence it represents $J_\nu(\alpha x) - x^\nu J_\nu(\alpha)$ in that interval, and our lemma is proved.

LEMMA 4. *If in the interval $0 \leqq x \leqq 1$, $\psi(x)$ is defined and such that $x\psi(x)$ has a Lebesgue integral in this interval, and if furthermore*

$$\int_0^1 x\psi(x)F_\nu(\lambda_n, x)dx = 0 \quad (n = 1, 2, 3, \cdots), \tag{23}$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the successive roots, positive or zero, of equation (3), *then $\psi(x)$ is zero at every point of the interval $0 < x < 1$ at which it is continuous; it is zero for $x = 1$ if continuous for that value and the l of equation* (3) $\neq 0$, *or if it is continuous at that point and at an infinite number of points in its neighborhood; it is zero for $x = 0$ if it is continuous at that point and at an infinite number of points in its neighborhood.*

The proof is somewhat different in the two cases where the l of equation (3) is different from or is equal to zero. We shall consider first the former case.

By Lemma 2 the function $J_\nu(\alpha x)$ may be expanded into a series of the form (13) that converges uniformly in the interval $0 \leqq x \leqq 1$. Multiplying the series by $x\psi(x)$ and integrating term by term* from 0 to 1, we obtain in view of (23)

$$\int_0^1 x\psi(x)J_\nu(\alpha x)dx = 0. \tag{24}$$

If in this equation we replace $J_\nu(\alpha x)$ by its development in a power series, integrate term by term,† and divide through by α^ν, we obtain

$$0 = \sum_{i=0}^{\infty} \frac{(-1)^i\alpha^{2i}}{2^{\nu+2i}i!\,\Gamma(\nu + i + 1)} \int_0^1 x^{\nu+1+2i}\psi(x)dx,$$

* It follows from the uniform convergence of the series for $J_\nu(\alpha x)$ and the fact that $x\psi(x)$ has a Lebesgue integral in the interval $0 \leqq x \leqq 1$ that we may obtain the value of the integral on the left hand side of (24) by integrating term by term.

† The justification is the same as before.

and this equation is known to hold for all real values of α except zero. Since both sides of the equation are analytic functions, it follows that the equation holds for all values of α and therefore that the right-hand side is identically zero. Hence we have

$$\int_0^1 x^{\nu+1+2i}\psi(x)dx = 0 \quad (i = 1, 2, 3, \cdots), \tag{25}$$

and therefore $\psi(x)$ is zero at every point of the interval $0 < x \leqq 1$ at which it is continuous,* and is zero for $x = 0$ if continuous at that point and at an infinite number of points in its neighborhood.

Let us now consider the case where the l of equation (3) is zero. Making use of Lemma 3 in the same way as we made use of Lemma 2 in the discussion of the previous case, we obtain in place of equation (25) the relation

$$\int_0^1 x^{\nu+1}(x^{2i} - 1)\psi(x)dx = 0 \quad (i = 1, 2, 3, \cdots).$$

From this we readily obtain

$$\int_0^1 x^{\nu+1+2i}(x^2 - 1)\psi(x)dx = 0 \quad (i = 0, 1, 2, \cdots).$$

Hence $\psi(x)$ is zero at every point of the interval $0 < x < 1$ at which it is continuous, and is zero for $x = 0$ or $x = 1$ if continuous at these points and at an infinite number of points in their respective neighborhoods. Thus our lemma is completely proved.

We are now ready to prove the following theorem.

THEOREM: *If the function $f(x)$ defined for the interval $0 \leqq x \leqq 1$ is such that $xf(x)$ has a Lebesgue integral in this interval, and if the series of the form (2) corresponding to $f(x)$ converges or is summable at all points of the interval $0 \leqq x \leqq 1$ at which $f(x)$ is continuous, defines a function $\varphi(x)$ that is continuous at those points† and such that $x\varphi(x)$ has a Lebesgue*

* Cf. Lerch, *Acta Mathematica*, vol. 27 (1903), p. 347. Also the writer, BULLETIN, vol. 14 (1908), p. 368 and vol. 15 (1908), p. 116. The proof of the theorem referred to is given in these papers for Riemann integrability; it applies equally well for Lebesgue integrability except at the point $x = 0$, where our added condition is necessary to complete the proof in the same way as it is completed in the writer's footnote on p. 371 of the first reference above.

† The point $x = 1$ is to be excepted for the case $l = 0$, unless $f(1) = 0$.

integral in the same interval, and can be integrated term by term over this interval after being multiplied through by a function that is continuous there, then the series will converge or be summable to $f(x)$ at all points of the interval $0 < x < 1$ at which $f(x)$ is continuous; it will converge or be summable to $f(x)$ for $x = 1$ if $f(x)$ is continuous there and $l \neq 0$, or if $f(x)$ is zero for $x = 1$, and is continuous at that point and at an infinite number of points in its neighborhood; it will converge or be summable to $f(x)$ for $x = 0$ if the function is continuous at that point and at an infinite number of points in its neighborhood.*

We have

$$\varphi(x) = \sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 xf(x)F_\nu(\lambda_n, x)dx}{\int_0^1 x[F_\nu(\lambda_n, x)]^2 dx}.$$

Multiplying by $xF_\nu(\lambda_n, x)$ for $n = 1, 2, 3, \cdots$ and integrating from 0 to 1, we obtain, in view of the fact that the F's form an orthogonal set,

$$\int_0^1 x[\varphi(x) - f(x)]F_\nu(\lambda_n, x)dx = 0 \quad (n = 1, 2, 3, \cdots).$$

From this equation and Lemma 4 our theorem follows at once.

UNIVERSITY OF CINCINNATI,
June, 1916.

SECOND NOTE ON REMOVABLE SINGULARITIES.

BY DR. W. E. MILNE.

IN a "Note on removable singularities"† the writer stated a theorem‡ concerning removable singularities for functions of several complex variables. An analogous theorem, with less restrictive hypotheses, is the following

* The convergence or summability to $\frac{1}{2}[f(x+0) + f(x-0)]$ at points where $f(x)$ has a finite jump does not follow directly from the present method. It can be obtained from the convergence or summability to $f(x)$ at points of continuity by a method given by the writer on pp. 428–429 of the paper in volume 10 of the *Transactions*, referred to above.

† BULLETIN, vol. 21 (1914), pp. 116–117.

‡ The proof there given for this theorem is incomplete, as Dr. Dunham Jackson pointed out. The gap is filled by the proof here given.

THEOREM. *Let the function* $\varphi(z_1, \cdots, z_n)$ *of* n *complex variables,* $n > 2$, *be analytic in the region* $(S) = (S_1, \cdots, S_n)$ *except for the points of a manifold* M *of such character that, when the* $n - 2$ *variables* $z_3, \cdots, z_n$ *are given any fixed values in their respective regions, the points of* M *whose last* $n - 2$ *coordinates are the above* $z_3, \cdots, z_n$ *are isolated, and thus their first two coordinates* z_1, z_2 *lead to isolated points in* (S_1, S_2). *Then the function* φ *has a limit in the points of* M, *and if defined as equal to its limit will be analytic without exception in* (S).

Proof. Let $P : (a_1, \cdots, a_n)$ be any point on M, and hold $z_3, \cdots, z_n$ fast at $(a_3, \cdots, a_n)$. Since the points in (S_1, S_2) corresponding to $(a_3, \cdots, a_n)$ (i. e., points (z_1, z_2) such that the points $(z_1, z_2, a_3, \cdots, a_n)$ are points of M) are isolated, it is possible to draw circles c_1 about a_1 and c_2 about a_2 small enough to exclude all the points thus arising except (a_1, a_2). Let t_1 be on c_1 and t_2 on c_2. Then φ will be analytic at $(t_1, t_2, a_3, \cdots, a_n)$ since this point is not on M, and hence to each (t_1, t_2) corresponds a positive number K such that $\varphi(t_1, t_2, z_3, \cdots, z_n)$ is analytic in $z_3, \cdots, z_n$ when $|z_i - a_i| < K$. Since the set of points (t_1, t_2) is closed it is easily seen that the lower limit K_0 of the corresponding set of K's is not zero, so that $\varphi(t_1, t_2, z_3, \cdots, z_n)$ is analytic in $z_3, \cdots, z_n$ when $|z_i - a_i| < K_0$, $i = 3, \cdots, n$, for *every* point (t_1, t_2) on c_1 and c_2.

Now φ may be so defined on M as to be analytic in z_1 and z_2 throughout (S_1, S_2) for every choice of $z_3, \cdots, z_n$ in $(S_3, \cdots, S_n)$, because the singularities in (S_1, S_2) are isolated, and isolated singularities for a function of two complex variables are removable. Therefore if z_1 is inside c_1 and z_2 inside c_2 we have

$$\varphi(z_1, \cdots, z_n) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_1}\int_{c_2} \frac{\varphi(t_1, t_2, z_3, \cdots, z_n)dt_1dt_2}{(t_1 - z_1)(t_2 - z_2)}.$$

The right-hand side of this equation is analytic in all n variables together at the point P, and therefore φ is analytic in all n variables at P. Since P was any point on M, the theorem is established.

Remark. Two important classes of singular manifolds to which the above theorem will apply were suggested to me by Professor Osgood.

The first is the class of manifolds defined in the neighborhood of a point by two equations

$$F_1(z_1, \cdots, z_n) = 0, \qquad F_2(z_1, \cdots, z_n) = 0,$$

where F_1 and F_2 are each analytic at the point in question and vanish there, but have no common factor there. Suppose for simplicity that the point is the origin. Then by Weierstrass's theorem of factorization it will be possible, at least after a non-singular linear transformation of the independent variables, to write

$$F_1 = [z_1^m + A_1 z_1^{m-1} + \cdots + A_m]\theta(z_1, \cdots, z_n),$$

$$F_2 = [z_1^\mu + B_1 z_1^{\mu-1} + \cdots + B_\mu]\psi(z_1, \cdots, z_n),$$

where the A's and the B's are functions of $z_2, \cdots, z_n$ which are analytic at the origin and vanish there, while θ and ψ are analytic at the origin and do not vanish there. The resultant of the polynomials in brackets is a function $R(z_2, \cdots, z_n)$ analytic at the origin and vanishing there, but not identically zero. Hence we can make a linear transformation of $z_2, \cdots, z_n$ so that $R(z_2, 0, \cdots, 0)$ is not identically zero. Now it will be seen that when $z_3, \cdots, z_n$ are all held fast at the origin the point $z_1 = 0, z_2 = 0, \cdots, z_n = 0$, is an isolated point of the manifold M. For in a sufficiently small neighborhood of the origin in the z_2-plane the function $R(z_2, 0, \cdots, 0)$ vanishes only when $z_2 = 0$; and when $z_2 = 0, z_3 = 0, \cdots, z_n = 0$, the only common solution of F_1 and F_2 in the neighborhood of the origin in the z_1-plane is $z_1 = 0$.

The second class is the class of $(2n - 4)$-dimensional real manifolds which are *regular* at the point in question (which we may again take as the origin). For then M is defined by the equations

$$z_i = \varphi_i(t_1, \cdots, t_m) + \sqrt{-1}\psi_i(t_1, \cdots, t_m),$$
$$(i = 1, 2, \cdots, n; \quad m = 2n - 4),$$

where the φ's and ψ's are single-valued real functions of the real parameters $t_1, \cdots, t_m$ which vanish at the origin and have continuous first partial derivatives there. Moreover the matrix

$$a = \left\| \begin{matrix} \dfrac{\partial \varphi_1}{\partial t_1} & \cdots & \dfrac{\partial \varphi_1}{\partial t_m} \\ \cdot & \cdot & \cdot \\ \dfrac{\partial \varphi_n}{\partial t_1} & \cdots & \dfrac{\partial \varphi_n}{\partial t_m} \\ \dfrac{\partial \psi_1}{\partial t_1} & \cdots & \dfrac{\partial \psi_1}{\partial t_m} \\ \cdot & \cdot & \cdot \\ \dfrac{\partial \psi_n}{\partial t_1} & \cdots & \dfrac{\partial \psi_n}{\partial t_m} \end{matrix} \right\|$$

is of rank m at the origin. A general homogeneous linear transformation of the z's,

$$T: \qquad z_i' = \sum_{j=1}^{n} a_{ij} z_j,$$

yields a linear transformation T' of the φ's and ψ's of the form

$$T': \quad \begin{aligned} \varphi_i' &= \sum_{j=1}^{n} (\alpha_{ij}\varphi_j - \beta_{ij}\psi_j), \\ \psi_i' &= \sum_{j=1}^{n} (\beta_{ij}\varphi_j + \alpha_{ij}\psi_j), \end{aligned}$$

where $a_{ij} = \alpha_{ij} + \sqrt{-1}\beta_{ij}$. If T' is non-singular then T is non-singular. Let δ be the matrix of the transformation T' and $\bar{a}$ the matrix corresponding to a after the transformation. Then

$$\bar{a} = \delta \times a.$$

The m-rowed determinants in $\bar{a}$ are polynomials in the α_{ij} and β_{ij} which are not identically zero, since a is of rank m. Therefore a particular transformation T' may be chosen so that all the m-rowed determinants in $\bar{a}$ are different from zero and also so that T' is non-singular when $t_1 = t_2 = \cdots = t_m = 0$. Then after the transformation T corresponding to T' has been made, if $z_3, \cdots, z_n$ are all held fast at the origin, the point $z_1 = 0$, $z_2 = 0$ is an isolated point of M.

BOWDOIN COLLEGE,
April, 1916.

THE MITTAG-LEFFLER TESTAMENT AND INSTITUTE.*

WE the undersigned, modifying our joint testament of January 6, 1883, do here declare our last will, which is to bequeath all our possessions, after our deaths, to a foundation bearing the name Mittag-Leffler Mathematical Institute.

This institute shall have for its purpose to maintain and develop still further the position now held by pure mathematics in the four Scandinavian countries of Sweden, Denmark, Finland, and Norway, but especially in Sweden, and also to make known and appreciated at its true value beyond their borders the share of these countries in the highest sphere of intellectual life.

In the accomplishment of this task, no objects other than those mentioned above shall be considered. No account shall be taken of personal friendships, or of the desire to lend pecuniary assistance to those in need. Neither shall any consideration be given to the needs of practical life, to examination questions, to political opinions, or to the aims of any science other than *pure* mathematics.

The Institute shall accomplish its task as follows:

1. By preserving and enriching the mathematical library of the undersigned G. Mittag-Leffler, with all that appertains to it in the way of manuscripts, portraits, family collections and souvenirs, and other objects.

The library shall remain in the large stone villa situated on our property in quarter No. 16, called Mitgård, at Djursholm, and shall not be incorporated in any other collection of books. The villa has been built and arranged with this purpose in view, and contains several rooms in which readers can work without disturbance.

The small part of the villa which is now used as a dwelling shall also be devoted, after our deaths, to the library.

The library shall be open to all mathematicians, but, to prevent abuses, only under the authorization of the president of the executive committee of the Institute. The books

* The following extract from the Mittag-Leffler testament was published on the occasion of the celebration at Stockholm of Dr. G. Mittag-Leffler's seventieth birthday, March 16, 1916. It is translated for the BULLETIN by Dr. CAROLINE E. SEELY.

shall not be lent out, and shall be used only in the library building.

2. By granting fellowships, for study at home or abroad, to young people of both sexes, belonging to the four countries above named, who have given evidence of real aptitude for research and discovery in the domain of *pure* mathematics.

Moreover, works of unusual importance, written by inhabitants of these countries, may be made the objects of special recognition, in the form of a gold medal, of the same form and design as the small Nobel medal, and as long as copies shall remain, of a set of the *Acta Mathematica,* in a special binding bearing the name of the author crowned.

3. By awarding prizes for discoveries really worthy of the name in the domain of *pure* mathematics. These prizes shall be given without regard to the nationality of the author. He may belong to any country whatever, and inhabitants of the four Scandinavian countries shall not enjoy, in this matter, any privilege. The prize shall be awarded only for a discovery bringing with it new ideas of such a nature that they give a new impetus to the science. It is desirable, however, that the prize be awarded once at least in every six years. This prize shall consist of a large gold medal and a diploma, of artistic design, the latter stating the scientific reasons for the award, and of as complete a set as possible of the *Acta Mathematica,* in special binding bearing the name of the recipient. He shall be invited to appear personally at Djursholm to receive the prize, a suitable appropriation for his travelling expenses being made. The prize shall be presented at a special ceremony in the great hall of the library.

4. When the annual revenues of the Institute shall exceed the amount indicated below, there may be created, beside the post of director, other remunerated positions, whose holders shall devote themselves to exclusively scientific writing and teaching in the domain of pure mathematics.

To the preceding provisions the following shall be added:

A. The Executive Committee of the Institute shall be composed of the Swedish members of the first class (pure mathematics) of the Royal Academy of Sciences, together with Professors Ivar Fredholm and N. E. Nörlund, during their lifetimes. The Director of the Institute shall also be a member. The Committee may also add to its membership, for a suitable period, any Swedish mathematician of real

eminence who shall agree entirely with the motives that have actuated us but who shall not yet have become a member of the first class of the Academy of Sciences. There may also be added to the Committee a mathematician from one of the three other Scandinavian countries, under the same conditions.

B. As soon as possible, there shall be appointed, to fill the post of scientific and executive Director of the Institute, a mathematician of eminent rank, who shall appear particularly qualified for this charge, and whose activities shall be devoted exclusively to personal research and shall contribute, at the same time, to the ends pursued by the Institute. He shall also, when suitable occasion arises and for strictly scientific purposes, give courses of lectures for a limited number of really gifted auditors.

From the material standpoint, he shall be placed in a more advantageous position than any professor of mathematics in any University in the four Scandinavian countries. He shall be domiciled at Djursholm, and if possible in the immediate neighborhood of the library. An appropriation shall be made for his house rent until a special building shall have been fitted up for his use. His nomination shall be made, on recommendation of the Committee, by His Majesty the King, if, as we venture to hope, His Majesty deigns to consent.

. . . . . .

E. At least once in every six years, the Institute shall hold a special meeting. The mathematicians of the four Scandinavian countries shall receive a personal invitation to attend. We hope that, in consideration of the importance of the Institute for these countries, all will be present if possible.

It is to be desired that the day of this meeting be chosen to coincide with the date of the meeting of the congress of Scandinavian mathematicians at Stockholm. On this occasion shall be read a report of the activities of the Institute since its last special meeting. The ceremony shall be invested with a character of dignity that shall emphasize plainly the elevated mission of the mathematical sciences, as well as the purpose of the activities of the Institute.

Finally, I, the undersigned G. Mittag-Leffler, wish to declare that the model which I have had in mind for the Institute founded by my wife and myself, is the Institut Pasteur at Paris. It seems to me that this Institute has fulfilled the

mission of a center of scientific research better than any university or academy. The universities have everywhere, in addition to their scientific task, another that often interferes very seriously with it, namely the training of teachers and officials. As for the academies, which have been the most effective institutions from the scientific standpoint up to the present time, they suffer from two disadvantages: on the one hand, the main activities of most of their members are not confined to their interests, and on the other hand, when this is not the case, the scientist lacks the stimulus to his own investigations to be found in guiding or assisting the researches of others. Our Institute is not connected with an establishment where experimental research can be carried on, but on the contrary, in conformity with the needs of pure mathematics, with a special and very complete library.

One may find in our country sufficient possibilities for the creation and organization of institutes for the natural sciences, but very few people, apart from specialists, understand the importance and the mission of pure mathematics. That is why the undersigned, G. Mittag-Leffler, has always desired to be able to found an institute like that which we hope to have established by this testament.

Our testament owes its origin to the lively conviction that a people that does not accord to mathematics a high place in its estimation, will never be in a position to fulfil the most lofty tasks of civilization, and to enjoy, in consequence, that international consideration which is itself, in the end, an effective means of preserving our place in the world and of safeguarding our right to live our own life.

The testament contains finally some directions to the effect that the Institute shall enter upon its activities on the death of G. Mittag-Leffler, except for certain life interests to be assured to Madame Signe Mittag-Leffler, and certain directions with regard to the administration of the estate and to certain small life interests and other allowances.

THE MADISON COLLOQUIUM LECTURES ON MATHEMATICS

Part II. Topics in the Theory of Functions of Several Complex Variables. By WILLIAM FOGG OSGOOD. New York, American Mathematical Society, 1914.

THE transition from the theory of functions of a single complex variable to the analogous theory for two variables corresponds in the domain of reals to a transition from functions on a plane to functions whose arguments are points of a four dimensional space. Generalizations from two to three real dimensions frequently involve great difficulties, even with the aid of our geometrical intuitions of the plane and three-space. It is not surprising therefore to find that familiar theorems concerning functions of a single complex variable either have no analogues at all, or else yield a multiplicity of generalizations, when the number of independent variables is increased. In the plane, for example, we have only the theory of connectivity associated with curves, while in space of four dimensions there are three kinds of connectivity corresponding in an analogous way to curves, surfaces, and three-spreads. On these subspaces of different dimensions there are theories of integrals corresponding to the curve integrals which play such a fundamental rôle in the theory of functions of a single complex variable.

The increase in complexity mentioned in the preceding paragraph arises from the greater multiplicity of geometrical configurations in the higher spaces. But there are also serious analytical obstacles to the extensions of the theory of functions of a single complex variable. Thus for a function $f(z_1, z_2)$ of two variables $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ the familiar potential equation is replaced by the system

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} = 0, \quad \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} = 0,$$

$$\frac{\partial^2 u}{\partial x_1 \partial y_2} - \frac{\partial^2 u}{\partial x_2 \partial y_1} = 0, \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial y_1 \partial y_2} = 0,$$

which admits of a much greater variety of initial conditions and is in other respects also more formidable.

A typical example of a theorem which breaks down entirely in the higher spaces is the following: Let $\varphi(x, y)$ be a function of two complex variables analytic at the origin and having $\varphi(0, y) \equiv 0$. With the help of the preparation theorem of Weierstrass φ can always be decomposed into a product of the form

$$\varphi(x, y) = P(x, y)M(x, y),$$

where $P(x, y)$ is a polynomial in y with coefficients which are series in x without constant terms, while M is a series in x and y with non-vanishing constant term. In a suitably chosen neighborhood of the origin, therefore, the function $y(x)$ defined by the equation $\varphi = 0$ is finitely multiple valued and has its values defined by the polynomial equation $P(x, y) = 0$. In the Colloquium Lectures Professor Osgood raises the question as to whether a similar theorem would be true for a function $y(x_1, x_2, \cdots, x_n)$ defined by an equation of the form

$$\varphi(x_1, x_2, \cdots, x_n, y) = 0. \tag{1}$$

He has more recently discovered that the theorem is not true when $n > 1$,* and the example by means of which he proves this statement provides some very interesting further information. An equation of the form (1) may in fact define a function $y(x_1, \cdots, x_n)$ which is single-valued and monogenic in a part of a neighborhood of the origin, while the origin itself is a point of a natural boundary of the function, other parts of the neighborhood being lacunary spaces. Such a function could clearly not be analytic or even algebroid at the origin.

The foregoing remarks will indicate to some extent the complexity of the questions which lie at the foundations of the theory of functions of several complex variables. Professor Osgood has been a thorough student of the subject with a keen appreciation of its deficiencies at the present stage of development. His Lectures seem to be a resumé preliminary to the publication of an exhaustive treatise, and it is much to be hoped that such a treatise may appear from his pen in the near future. Lecture I contains a historical introduction and "general survey" of the field, while the other four lectures are devoted to a descriptive and critical exposition of fundamental theorems, some of the most sig-

* *Transactions Amer. Math. Society*, vol. 17 (1916), p. 4.

nificant of which are the result of the author's own researches. The whole will be most helpful to students of the subject.

In the first lecture the author mentions a definition of a function of several complex variables published by Cauchy in 1831, but the earliest phases of the theory to be studied in detail were associated with special problems. Among the more important of these were the extensions of the existence theorems of Cauchy to partial differential equations and systems of equations defining implicit functions of several complex variables, the periodic abelian functions arising from Jacobi's problem of inversion, and periodic functions of several variables in general. The existence theorems did not require the development of new methods, their extensions being obtained by means of the famous device of dominating series applied by Cauchy to the simpler cases. The masterly generalization of the elliptic inversion problem by Jacobi was, however, the precursor of a long series of investigations during one of the most fruitful periods in the annals of mathematics. It was during this period that the theory of the theta functions of several variables, which lies at the heart of the periodic function theory, was so highly developed by Riemann and Weierstrass.

The results which have just been mentioned were the products of successful attempts to generalize the elliptic functions and elliptic theta functions. After the development of the theory of automorphic functions of a single variable by Klein and Poincaré in the early eighties, it was natural that similar generalizations should be sought for functions of several variables. Picard was the pioneer in this field. The functions studied when, for example, there are two independent complex variables x, y are those which are invariant under groups of projective transformations, or groups of transformations of the form

$$x' = \frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, \quad y' = \frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2}.$$

The theory of these functions in turn stimulated the interest of Picard and his associates in algebraic functions of two variables and the surface and line integrals of various kinds associated with them. Their results are analogous in a general way to those of the theory of algebraic functions of a single variable, although the theorems in particular cases are strikingly different.

Lecture II is devoted to a rather heterogeneous collection of theorems. It begins with an abbreviated account of a generalization of Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t - z}$$

and the theory of residues. Several sections are then devoted to a discussion of theorems giving sufficient conditions that a function of several complex variables be analytic, rational, or algebraic. Weierstrass stated the theorem that a function $f(z_1, z_2, \cdots, z_n)$ of several complex variables is surely rational if it is representable near every point of analysis by the quotient of two functions analytic and prime to each other at the point in question. He did not specify explicitly the character of the infinite region implied in the phrase "at every point of analysis," but he seems to have intended that each of the variables $z_1, z_2, \cdots, z_n$ should range independently over a plane extended by a single point at infinity. Professor Osgood has called attention to the fact that the part of the hypothesis of the theorem referring to the infinite region is under these circumstances equivalent to presupposing properties of the function f at finite points into which the points of the infinite region have been converted by a transformation of the form

$$z_k' = \frac{\alpha_k z_k + \beta_k}{\gamma_k z_k + \delta_k}, \qquad (k = 1, 2, \cdots, n).$$

It is a theorem due to himself that a conclusion similar to that of Weierstrass can be drawn if the transformation just written is replaced by a projective transformation on the n variables z. This involves a conception of the points at infinity different from that of Weierstrass, and there are gradations between the two, with possibly similar theorems, if the variables are transformed one part projectively and the rest bilinearly. In Section 6 of this lecture Professor Osgood draws the very interesting conclusion that similar results can be obtained by assuming properties of the function f at a suitably characterized portion only of the region at infinity.

Let a power series in x, y be convergent for $|x| < r$, $|y| < s$, and let s be the largest positive constant for which, when r is fixed, the series has this convergence property. Then s is

a function $s = \varphi(r)$ called the radius of convergence associated with r. Sections 7 and 8 of this lecture contain a summary of the known properties of the function $\varphi(r)$ and theorems related to it.

The last part of Lecture II describes generalizations of the famous theorems of Weierstrass and Mittag-Leffler concerning the construction of a function of a single complex variable having assigned poles and zeros in a given continuum. There always exists such a function analytic except at the poles, expressible as a quotient of two other functions each analytic at every point of the continuum. The essence of the generalization to functions of several complex variables lies in the characterization of the points which correspond to poles or zeros. Expressed very roughly the theorem is that if at every point $a = (a_1, a_2, \cdots, a_n)$ of the space a quotient

$$f_{(a)}(z_1, z_2, \cdots, z_n) = \frac{H_{(a)}(z_1, z_2, \cdots, z_n)}{G_{(a)}(z_1, z_2, \cdots, z_n)} \tag{2}$$

is assigned, where H and G are analytic and prime to each other at the point in question, then there exists a quotient $f = H/G$ of two permanently convergent power series which is equivalent at every point a to the corresponding quotient $f_{(a)}$. Two functions are equivalent at a point a if their quotients, taken both ways, have only removable singularities in a neighborhood $T_{(a)}$ of the point. Professor Osgood describes also a more general theorem of Cousin, and some consequences involving generalizations by Gronwall of the Weierstrassian notion of prime factors of integral functions.

The third lecture is concerned with singular points and analytic continuation. A non-essential singularity of the first kind, sometimes called a pole, is a point $(a_1, a_2, \cdots, a_n)$ at which the function is expressible in the form (2) with $H_{(a)} \neq 0$, $G_{(a)} = 0$ at $(a_1, a_2, \cdots, a_n)$; the point is a non-essential singularity of the second kind if $H_{(a)}$ and $G_{(a)}$ both vanish but have no vanishing factors in common there; all other non-removable singularities are essential.

Professor Osgood gives a number of theorems, by Kistler and Hartogs, specifying conditions under which a singularity can be removed after assigning suitably chosen values to the function at the point itself and at other exceptional points near it. No non-removable singularities are isolated when $n > 1$, and in particular every essential singularity is a limit

point of others of the same type. Starting from a theorem concerning analytic continuation Professor Osgood shows how Hartogs has proved that if a function $f(x, y)$ of two complex variables is analytic except at the points of a two-dimensional continuous locus $y = \varphi(x)$ consisting of non-removable singularities, then $\varphi(x)$ itself must be analytic. This is a sharp contrast to the fact that the singularities of a function of a single complex variable may have a distribution as arbitrary as the boundary of an arbitrarily selected two-dimensional continuum.

Another striking theorem, deducible from a theorem of Levi which Professor Osgood characterizes as one of the most important contributions to the theory in recent years, is that a function $f(x, y)$ analytic (or meromorphic) at all points of the boundary of a finite and regular four-dimensional region T, can always be continued analytically (or meromorphically) throughout T. This has as a consequence that such a function can not have a finite lacunary region around which it is analytic. A three-dimensional locus

$$\varphi(x_1, x_2, y_1, y_2) = 0,$$

where $x = x_1 + ix_2$, $y = y_1 + iy_2$, can not be arbitrarily selected if it is to be a natural boundary for a function $f(x, y)$; it must satisfy a certain differential inequality. Levi-Civita has shown further that a function of two complex variables is in general uniquely determined if its values are assigned along a two-dimensional locus. But there are exceptional "characteristic" loci along which arbitrarily assigned values may determine none or an infinity of functions, a situation which reminds us forcibly that we are dealing more or less directly with initial values for solutions of a system of partial differential equations.

In Lecture IV Professor Osgood discusses certain questions in implicit function theory based upon the preparation theorem of Weierstrass. This theorem states that, a function $F(u; x_1, x_2, \cdots, x_n)$ analytic at the origin and having a lowest term in the series $F(u; 0, 0, \cdots, 0)$ of order m, is expressible as a product

$$F = P\Omega, \tag{3}$$

where Ω is a series in $u, x_1, \cdots, x_n$ with constant term different from zero, while P is a polynomial in u of degree m with

leading coefficient unity, its other coefficients being series in $x_1, \cdots, x_n$ vanishing at the origin. Near the origin the roots of F are clearly all roots of the polynomial P, and vice versa. The theorem mentioned in an earlier paragraph, which Professor Osgood stated tentatively in the Lectures, and later disproved by an example, would have implied a generalization of this including the case when $F(u; 0, 0, \cdots, 0)$ vanishes identically.

The preparation theorem gives information with respect to the roots of a single equation $F = 0$ in a neighborhood of the origin, which is completely satisfactory for many purposes. For a system of equations

$$\Phi_k(u_1, \cdots, u_l; x_1, \cdots, x_p) = 0 \quad (k = 1, 2, \cdots, l), \tag{4}$$

where the functions Φ_k are also analytic and vanish at the origin, Weierstrass has given a further important theorem, described in Section 6 of Lecture IV. He was interested in characterizing the manifold of $(l + p)$-dimensional points, rather than in the implicit functions u of the variables x defined by equations (4).

The relation between F and the polynomial P in equation (3) could equally well have been expressed in the form $P = MF$, where $M = 1/\Omega$ is a series with constant term different from zero. In a paper published in 1912* the author of this review has shown that when the homogeneous polynomials of lowest degree, the "characteristic" polynomials, of the series $\Phi_k(u_1, \cdots, u_l; 0, \cdots, 0)$, have a resultant R different from zero, there always exists a polynomial $P(u_l; x_1, \cdots, x_p)$ expressible linearly in the form

$$P = M_1\Phi_1 + M_2\Phi_2 + \cdots + M_l\Phi_l,$$

and such that every root system $(u_1, \cdots, u_l; x_1, \cdots, x_p)$ of equations (4) near the origin has values $(u_l; x_1, \cdots, x_p)$ making $P = 0$; while every sufficiently small root system of $P = 0$ is part of at least one solution of these equations near the origin in the $(l + p)$-dimensional space. The degree N of P in u_l is the product of the degrees of the characteristic polynomials. Further when the substitution

* A generalization of Weierstrass's preparation theorem for a power series in several variables, *Transactions Amer. Math. Society*, vol. 13 (1912), p. 133.

$$(5) \qquad u_l = -t_1u_1 - t_2u_2 - \cdots - t_{l-1}u_{l-1} - z$$

is made in equations (4), the polynomial corresponding to P has the form

$$Q(z;\, x_1, \cdots, x_p) = \prod_{i=1}^{N} (z - t_1u_{i1} - t_2u_{i2} - \cdots - t_{l-1}u_{i(l-1)} - u_{il}),$$

where the systems $(u_{i1}, \cdots, u_{il};\, x_1, \cdots, x_p)$ are the root systems of (4) corresponding to $x_1, \cdots, x_p$. The introduction of the variables t is a well-known device of the algebraic elimination theory serving to unite the u-values which belong to a single root system of the equations.

It seems to me that the theorems described by Professor Osgood in Section 7 of Lecture IV could be deduced at once from the properties of the polynomial Q. For example, since Q is of the Nth degree, the system (4) must have exactly N root systems near the ux-origin for each point $(x_1, \cdots, x_p)$ in a properly chosen neighborhood of the origin in the x-space. These systems will all be distinct unless $(x_1, \cdots, x_p)$ makes the z-discriminant of Q identically zero in the variables t. The coefficients of the various t-terms are, however, series in $x_1, \cdots, x_p$ which set equal to zero define a locus analogous to Professor Osgood's "locus D," but not, if I understand him correctly, exactly the same. At every point of this locus some of the root systems of equations (4) coincide. Professor Osgood has stated his results for a more general hypothesis than mine. He requires only that the series $\Phi_k(u_1, \cdots, u_l;\, 0, \cdots, 0)$ shall have no common root near the origin except that point itself. The theorems of my paper apply, however, under very general circumstances, since the degrees of the characteristic polynomials are entirely arbitrary and their coefficients subjected to an inequality only. I must confess that my proofs are complicated, and I hope that this undesirable feature will be eliminated when Professor Osgood publishes in detail the theorems in his more general form.

On page 196 of the Lectures there is a reference to my paper which might lead one to infer that its character is similar to that of a paper by MacMillan. The hypotheses of the two papers are the same, but their principal theorems are entirely different. MacMillan discusses the highly interesting theorem, originally stated by Poincaré, that the solu-

tions of the equations (4) near the origin are the same as those of another system whose first members are polynomials in the variables u. The polynomials have in general extraneous roots which are not necessarily small. The polynomial Q described above defining the solutions near the origin would be found from them by eliminating the u's algebraically and applying the preparation theorem of Weierstrass to the resulting polynomial in z.

Lecture V describes a point of view in the algebraic function theory which seems most interesting and effective. Let $f(w, z) = 0$ be an irreducible algebraic equation of deficiency p, having a corresponding Riemann surface F' rendered simply connected by period cuts. When $p = 1$ the equation $t = w(z)$, where w is the integral of the first kind, maps F' upon a curvilinear parallelogram in the t-plane, and the theory of functions and integrals on the Riemann surface then corresponds completely to that of a system of elliptic and allied functions with specified periods and t as independent variable. The Weierstrassian function $\sigma(t)$ is the fundamental function for such a system, in terms of which all the others may be expressed. The theory is more uniform in the t-plane than on the Riemann surface, because the particular t-values which correspond to branch points of the surface are no more exceptional than other values of t.

For deficiencies $p > 1$ Klein has defined a "prime function" on the Riemann surface which plays a rôle analogous to that of the σ-function for the simpler case. Professor Osgood proposes to transfer the theory of this function to a t-plane on which the surface is mapped by a transformation $z = \varphi(t)$, the function φ being a suitably selected automorphic function. He shows concisely but clearly in this lecture how the various types of functions analogous to those on the Riemann surface may be expressed in terms of the prime function $\Omega(t, \tau)$, and points out the advantages, in many cases decisive, to be gained by this uniformization.

It is not to be expected that the complications of a structure like that of the algebraic function theory can be entirely eliminated by a process such as Professor Osgood would have us use. But it is possible, on the other hand, that they may be so localized as not to mar the more facile beauty of the other parts of the theory. A first difficulty from his point of view seems to lie in the proof of the existence of the uni-

formizing automorphic function $\varphi(t)$, a proof which he has treated in the second edition of his Funktionentheorie with detail. The prime function is defined in Lecture V as a limit

$$\Omega(t, \tau) = \lim_{\Delta t=0,\ \Delta\tau=0} \sqrt{\Delta t \Delta \tau e^{-\Pi_{t\tau}^{t+\Delta t,\ \tau+\Delta\tau}}},$$

where $\Pi_{t'\tau'}^{t\tau}$ is a function analogous to the difference

$$P_{\xi\eta}^{xy} = P_{\xi\eta}(x) - P_{\xi\eta}(y)$$

formed for a suitably chosen integral $P_{\xi\eta}(x)$ of the third kind on the Riemann surface. It is seen that the definition involves a function of four arguments, and it is on account of this circumstance that the fifth lecture is a natural continuation of the preceding four. For the theorems of Lecture II, in particular, describing the analytic character of a function of several simultaneous variables resulting from known properties of the function when all but one variable are considered as parameters, play an important and apparently indispensable rôle in establishing the existence and properties of the functions Π and Ω.

GILBERT AMES BLISS.

THE UNIVERSITY OF CHICAGO.

SHORTER NOTICE.

Mathematiker-Anekdoten. By W. AHRENS. Band XVIII of the Mathematische Bibliothek edited by Lietzmann and Witting. Leipzig, B. G. Teubner, 1916. Pp. 56. Price 80 pfennige.

ANYTHING from the pen of Dr. Ahrens is always sure to be interesting. His Mathematische Unterhaltungen und Spiele, his Mathematische Spiele, and his Scherz und Ernst in der Mathematik are all well known and are everywhere appreciated. He is one of those rare writers on the bizarre in mathematics who keep their balance and turn out works which have the stamp of dignity and learning, while at the same time revealing the lighter side of the science. In other words, he is a man who evidently combines in his own soul the Scherz

und Ernst, as he combines these elements in the title of one of his best known works.

On this account the reader is justified in expecting something worth his attention in a book of Mathematiker-Anekdoten compiled by such a writer; and if, by chance, he is a little disappointed in the diminutive size of the work under review, he may well ask himself what he expects for 80 pfennige. Surely, one cannot look for a seven-course intellectual dinner for 20 cents.

To American and English readers the chief interest in the anecdotes lies in the fact that most of them are not to be found in works in our language. The anecdotes in themselves are not numerous, nor are they, on the whole, as interesting as those to be found in some of Rebière's curious collections, but they are not of the stereotyped variety and hence they are welcome.

The first in Dr. Ahrens's list is Adam Riese, known to most Germans from the phrase "nach Adam Riese," which led, no doubt, to "according to Cocker" in England and "according to Daboll" in this country. The story is not of particular interest, but it lightens up, in a way, the general impression of Riese which comes to one from reading his works. The next one in the list is Gauss, concerning whose school days a couple of interesting stories are told; how he, at the age of nine, saw at a glance how to sum an arithmetic series, and how, at fifteen, he was reading Newton, Euler, and Lagrange.

Among others concerning whom Dr. Ahrens relates various anecdotes are the mathematical prodigy Henri Mondeux, with his examination by Cauchy; Riemann, "einer der genialsten und tiefsinnigsten Mathematiker aller Zeiten;" Grassmann, whom the vocational-guidance extremists would quite certainly have turned from the paths of mathematics in his younger days; Schellbach, whose nature the poet Novalis described, to the mind of Dr. Ahrens, in the statement that "Das Leben der Götter ist Mathematik. Alle göttlichen Gesandten müssen Mathematiker sein. Reine Mathematik ist Religion. Die Mathematiker sind die einzig Glücklichen. Der echte Mathematiker ist Enthusiast per se. Ohne Enthusiasmus keine Mathematik;" Ohm the physicist; August; Lasswitz the poet, who was professor of mathematics in the Gymnasium at Gotha; Euler; and Arago, with his interesting examination by Monge and Legendre. There are also notes

on certain interesting newspaper problems, a problem of exchange under war conditions, an incident in connection with Kästner's epigram on logarithms, and, finally, the Fermat problem.

All this is told in a style that is in harmony with the nature of the work and that would repay the reader even if the anecdotes themselves did not seem worth his time, which is far from being the case.

DAVID EUGENE SMITH.

NOTES.

THE July number (volume 17, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "On a general class of series of the form $\Sigma c_n g(x+n)$," by R. D. CARMICHAEL; "The geometries associated with a certain system of Cremona groups," by J. W. YOUNG and F. M. MORGAN; "A reduction of certain analytic differential equations to differential equations of an algebraic type," by W. D. MACMILLAN; "A new canonical form of the elliptic integral," by BESSIE I. MILLER; "On the notion of summability for the limit of a function of a continuous variable," by L. L. SILVERMAN; "On the factorization of Cremona plane transformations," by J. W. ALEXANDER; "Weierstrass's non-differentiable function," by G. H. HARDY; "Finite groups represented by special matrices," by G. A. MILLER; "On infinite regions," by W. F. OSGOOD; "Point sets and allied Cremona groups (Part II)," by A. B. COBLE; "Infinite products of analytic matrices," by G. D. BIRKHOFF.

THE July number (volume 38, number 3) of the *American Journal of Mathematics* contains the following papers: "A class of asymptotic orbits in the problem of three bodies," by L. A. H. WARREN; "Some invariants of the ternary quartic," by H. I. THOMSEN; "Functions of surfaces with exceptional points or curves," by C. A. FISCHER; "Dupin's cyclide as a self-dual surface," by MABEL M. YOUNG; "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface. Second memoir," by G. M. GREEN; "The asymptotic equation and satellite conic of the plane quartic," by TERESA COHEN.

THE June number (series 2, volume 17, number 4) of the *Annals of Mathematics* contains the following papers: "On the Wronskian test for linear dependence," by MAXIME BÔCHER; "Note on a theorem on envelopes," by W. R. LONGLEY; "Non-essential singularities of functions of several complex variables," by DUNHAM JACKSON; "A congruence of circles," by F. W. BEAL; "A case of iteration in several variables," by A. A. BENNETT; "The arithmetic genus of an algebraic manifold immersed in another," by S. LEFSCHETZ; "A characteristic property of self-projective curves," by L. L. DINES.

THE September number (volume 18, number 1) of the *Annals of Mathematics* contains: "On the surface of lowest degree passing through a given curve in space," by T. HAYASHI; "A practical method of determining elementary divisors," by H. T. BURGESS; "Conjugate systems with equal point invariants," by L. P. EISENHART; "On the derivatives of a function at a point," by J. F. RITT; "An existence theorem for the solution of a type of real mixed difference equation," by A. A. BENNETT; "Double elliptic geometry in terms of point and order alone," by J. R. KLINE; "An application of a group of order 16 to a configuration on an elliptic curve," by A. EMCH.

AT the meeting of the Edinburgh mathematical society on June 9 the following papers were read: By E. T. WHITTAKER: "On the solution of Riccati's equation by continued fractions"; by H. DATTA: "On symmetric determinants and Pfaffians"; by A. B. JEFFERY: "Bipolar and toroidal coordinates"; by W. P. MILNE: "The polars of the Hessian of a cubic curve" and "Determinantal systems of points"; by F. G. TAYLOR: "An involution pencil of whole-plane birationally related cubics."

A meeting of mathematicians from Sweden, Denmark, Finland, and Norway was held at Stockholm August 30–September 2. It will be recalled that the Sixth International Congress of Mathematicians was to have met at Stockholm at that time.

THE firm of John Wiley and Sons announces the publication of Lectures on Ten British Mathematicians of the Nineteenth Century, by the late ALEXANDER MACFARLANE.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1916–1917:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mechanics of continuous bodies, especially of viscous fluids and elastic bodies, three hours.—By Professor L. DONATI: Modern electromagnetic theories, with reference to recent researches of H. Kamerlingh Onnes and A. Righi, three hours.—By Professor F. ENRIQUES: Theory of singularities of algebraic functions, three hours.—By Professor S. PINCHERLE: Theory of sets; theory of functions of a real variable; the modern meaning of definite integration; Fourier's series, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Introduction to the general theories of electricity and magnetism, four hours.—By Professor G. PENNACCHIETTI: Hydrodynamics, four hours.—By Professor G. SCORZA: Abelian functions with geometrical applications, three hours.—By Professor C. SEVERINI: Advanced part of differential and integral calculus, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Functions of one and several complex variables, four hours.—By Professor G. LORIA: Theory of algebraic surfaces, especially cubics and quartics, three hours.—By Professor O. TEDONE: Principles of the mechanics of electrons, three hours.

UNIVERSITY OF MESSINA.—By Professor P. CALAPSO: Functions of a complex variable; elliptic functions, four hours. —By Professor G. Z. GIAMBELLI: Hyperspaces; moduli of algebraic forms; forms in several series of variables, four hours. —By Professor E. LAURA: Electromagnetic theory of light, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics from Galileo to Newton, three hours.—By Professor A. DEL RE: Analysis of Grassmann and theoretical astronomy, four and one half hours.—By Professor R. MARCOLONGO: Mathematical theory of vibrations of isotropic elastic bodies, three hours.—By Professor D. MONTESANO:

Geometry of the space of conics, three hours.—By Professor E. PASCAL: Selected topics of mathematical analysis, three hours.—By Professor L. PINTO: Electro-optics; hertzian waves, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Harmonic and polyharmonic functions; analytic functions; integral equations, four hours.—By Professor P. GAZZANIGA: Theory of numbers, four hours.—By Professor T. LEVI-CIVITA: General principles of analytic mechanics; newtonian potential; relativity; Einstein's gravitational statics, four hours.—By Professor G. RICCI: Absolute differential calculus; elasticity, four hours.—By Professor F. SEVERI: Algebraic geometry, four hours.—By Professor A. TONOLO: Partial differential equations of the first order, three hours.—By Professor G. VERONESE: Foundations of geometry, three hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Theory of complex variables; elliptic and modular functions, three hours.—By Professor M. DE FRANCHIS: Jacobi's inversion problem and Picard's varieties; hyperelliptic surfaces, three hours.—By Professor M. GEBBIA: Mechanics of continuous media; newtonian potential; hydrostatics and hydrodynamics, four and one half hours.—By ————: Mechanics (advanced part), three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: General theory of algebraic forms with geometrical applications, three hours.—By Professor U. CISOTTI: Ancient and modern analytical methods of mathematical physics and their most conspicuous applications, three hours.—By Professor F. GERBALDI: Functions of a complex variable; elliptic functions, three hours.—By Professor G. VIVANTI: Theory of partial differential equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: The geometry on an algebraic curve with transcendental method, three hours.—By Professor L. BIANCHI: Theory of functions of a complex variable; elliptic functions, four and one half hours.—By Professor U. DINI: General theory of series, particularly of divergent ones; analytic representation of

function in real and complex fields, four and one half hours.—By Professor G. A. MAGGI: The potential function and its direct applications; equilibrium and motion of elastic bodies, four and one half hours.—By Professor P. PIZZETTI: Introduction to spherical astronomy and to determination of elliptic orbits; mechanical theory of the shape and of the motion of rotation of the planets, four and one half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Differential geometry, three hours.—By Professor U. CRUDELI: Functions of lines, three hours.—By Professor L. SILLA: Kinematics and mechanisms, three hours.—By Professor V. VOLTERRA: Selected problems of mathematical physics, three hours; Hydrodynamics and aerodynamics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Potential; selected topics of analytical mechanics, three hours.—By Professor G. FUBINI: Cantor's numbers; entire and algebraic entire numbers; theory of numbers and forms with algebraic applications; applications of analysis to the theory of numbers, three hours.—By Professor C. SEGRE: Higher views concerning elementary geometry, three hours.—By Professor C. SOMIGLIANA: Electromagnetism with special regard to propagation phenomena, three hours.

PROFESSOR H. HAHN, of the University of Czernowitz, has been appointed professor of mathematics at the University of Bonn.

PROFESSOR C. JUEL has retired from the editorial board of the *Nyt Tidsskrift for Matematik*. Professor P. HEEGAARD, of the University of Copenhagen, is his successor.

PROFESSOR E. ZERMELO, of the University of Zurich, has retired. Professor R. FUETER, of the technical school at Carlsruhe, has been appointed his successor.

PROFESSOR G. SCORZA, of the University of Parma, has been promoted to a full professorship of projective and descriptive geometry, and transferred to the University of Catania for the current year.

THE Italian society of sciences (the forty) has awarded its gold medal for 1915 to Professor P. CALAPSO, of the University of Messina, for his researches in geometry, especially in the extensional deformation of quadrics.

PROFESSOR F. GERBALDI, of the University of Pavia, has been elected to membership in the Royal institute of Lombardy.

DURING the months of May and June, Professor J. HADAMARD, of the Ecole Polytechnique, gave a series of lectures on mathematical analysis at the Universities of Bologna, Genoa, and Rome.

DR. J. W. L. GLAISHER, of Trinity College, Cambridge, has been elected a member of the Royal society of Edinburgh.

HON. BERTRAND RUSSELL, former fellow of Trinity College, Cambridge, will be unable to lecture at Harvard University during the present year.

AT Vassar College, Dr. ELIZABETH B. COWLEY has been promoted to an associate professorship of mathematics.

DR. C. L. E. MOORE, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

AT the Rice Institute, Professor G. C. EVANS has been promoted to a full professorship of pure mathematics, and Dr. W. C. GRAUSTEIN to an assistant professorship of mathematics. Dr. F. D. MURNAGHAN, of Johns Hopkins, has been appointed to an instructorship in mathematics, and Mr. H. E. BRAY, of Harvard University, has been appointed to one of the two graduate fellowships in mathematics.

DR. A. A. BENNETT, of Princeton University, has been appointed adjunct professor of mathematics at the University of Texas.

PROFESSOR C. N. HASKINS, of Dartmouth College, has been promoted to a full professorship of mathematics.

MR. W. C. EELLS has resigned his position in the department of mathematics at the United States Naval Academy to accept an appointment as professor of applied mathematics at Whitman College, Walla Walla, Wash.

DR. D. D. LEIB, instructor in mathematics at the Sheffield Scientific School, Yale University, has been appointed assistant professor of mathematics and physics at the Connecticut College for Women.

AT the University of Kansas, Dr. S. LEFSCHETZ and Mr. J. J. WHEELER have been promoted to assistant professorships and Mr. E. B. MILLER, of the University of Chicago, has been appointed to an instructorship in mathematics.

AT Northwestern University, Dr. C. N. YEATON has been promoted to an assistant professorship of mathematics. Dr. C. E. WILDER, of Pennsylvania State College, Dr. F. E. WOOD, and Mr. I. ROMAN have been appointed instructors in mathematics.

DR. OTTO DUNKEL, of the University of Missouri, has been appointed assistant professor of mathematics at Washington University, St. Louis.

DR. J. E. ROWE, of Pennsylvania State College, has been promoted from an assistant to an associate professorship of mathematics.

DR. R. L. BORGER, of the University of Illinois, has been appointed professor of mathematics at Ohio University, Athens, Ohio.

DR. G. H. LIGHT has been appointed instructor in mathematics at the University of Colorado.

DR. J. V. DE PORTE, of Cornell University, has been appointed instructor in mathematics at the State college for teachers, Albany.

AT Syracuse University, Dr. J. L. JONES, of the University of Pittsburgh, has been appointed instructor in mathematics.

At the University of Nebraska, Mr. A. Babbett and Miss M. A. Colpitts have been appointed instructors in mathematics.

Dr. N. Altshiller, of the University of Colorado, has been appointed instructor in mathematics at the University of Oklahoma.

Dr. S. P. Sousley, of St. Mary's College, Emmetsburg, Maryland, has been appointed instructor in mathematics at Pennsylvania State College.

Professor E. Mach, of the University of Vienna, died February 22, 1916, at the age of 78 years.

Dr. K. Schwarzschild, director of the observatory at Potsdam, died May 11, 1916, at the age of 42 years.

Professor W. Vogt, of the University of Heidelberg, was killed in battle March 3, 1916, at the age of 32 years.

Don Jose Echegaray, professor of mathematical physics in the University of Madrid, and distinguished also as a poet and dramatic author, died September 15, 1916, at the age of eighty-three years. Professor Echegaray was a member of the academy of science of Madrid, which celebrated the fiftieth anniversary of his admission to the academy on March 12, 1916.

Edgar H. Harper, professor of mathematical physics in University College, Cork, known for his work on aviation, was killed while serving as a lieutenant.

Dr. G. A. Hill, formerly professor of physics at Harvard University, and author of a number of text books in physics and mathematics, died August 17, 1916, at the age of 74 years.

Professor J. P. D. John, formerly professor of mathematics at De Pauw University, died August 7, 1916, at the age of 73 years.

Professor W. C. Esty, of Amherst College, died July 27, 1916, at the age of 78 years. Professor Esty was Walker

professor of mathematics from 1865 to his retirement from active service in 1905.

DR. EMORY MCCLINTOCK died July 10 at the age of seventy-six years. Dr. McClintock was the first president of the American Mathematical Society, serving in that office from 1890 to 1894. He graduated from Columbia College in 1859 and was assistant professor of mathematics at Columbia in 1859 and 1860. He was actuary of the Asbury Life Insurance Company from 1867 to 1871, and of the Northwestern Mutual Life Insurance Company from 1871 to 1888, when he became connected with the Mutual Life Insurance Company of New York, of which he was also vice-president from 1905 to his retirement in 1911. He was a frequent contributor to the *American Journal of Mathematics* and other journals, and was one of the founders of the *Transactions* of the American Mathematical Society.

FREDERICK W. FRANKLAND died in New York City on July 24, in the sixty-second year of his age. He was a Fellow of the Institute of Actuaries, and a member of the American Mathematical Society since 1893. He was for several years government examiner in mathematics and insurance in New Zealand, and was author of numerous papers on mathematical, metaphysical, and sociological subjects.

JOSIAH ROYCE, Alvord professor of philosophy at Harvard University, died September 14, 1916, at the age of sixty years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABENBÉDER. See PÉREZ (J. A. S.).

BENEDICT (S. R.). A comparative study of the early treatises introducing into Europe the Hindu art of reckoning. (Diss., University of Michigan.) Concord, N. H., Rumford Press, 1916. 6+126 pp.

BIANCHI (L.). Lezioni sulla teoria delle funzioni analytiche di variabile complessa e delle funzioni ellittiche. 2a edizione. Pisa, E. Spoerri (F. Mariotti), 1916. 8vo. 685 pp. L. 25.00

CASHMORE (M.). Fermat's last theorem. London, Bell, 1916. 63 pp. 2s.

COOLIDGE (J. L.). A treatise on the circle and the sphere. Oxford, Clarendon Press, 1916. 602 pp. 21s.

GREGORY (R. A.). Discovery: or, the spirit and service of science London, Macmillan, 1916. 10+340 pp. 5s.

JESSOP (C. M.). Quartic surfaces with singular points. Cambridge, University Press, 1916. 35+197 pp. 12s.

LEUDET (M.). See PETIT (G.).

LÖVGREN (B. N.). Om hela tals delbahet med hela tal. Stockholm, A. Bonniers Verlag, 1915. 30 pp. Kr. 1.50

MACFARLANE (A.). Lectures on ten British mathematicians of the nineteenth century. New York, Wiley, 1916. 8vo. 148 pp. Cloth. $1.25

PÉREZ (J. A. S.). Compendio de álgebra de Abenbéder. Texto árabe, traducción y estudio. Madrid, E. Maestre, 1916. 8vo. 48+117+78 pp. Pes. 6.

PETIT (G.) et LEUDET (M.). Les allemands et la science. Paris, Alcan, 1916. 20+375 pp. Fr. 3.50

SÖDERBLUM (A.). Inledning till rymdens analytiska geometri. Gotenburg, Gumperts Verlag, 1915. 65 pp. Kr. 2.50

TOGLIATTI (E. G.). Le superficie di sesto ordine con infinite coniche. Milano, Hoepli, 1916. 4to. 64 pp. L. 3.85

VOLTERRA (V.). Teoria delle potenze dei logaritmi e delle funzioni di composizione. Roma, tip. R. Accademia dei Lincei, 1916. 4to. Pp. 167–249.

ZORETTI (L.). Exercices numériques et graphiques de mathématiques sur les Leçons de mathématiques générales du même auteur. Paris, Gauthier-Villars, 1916. 8vo. 16+124 pp. Fr. 7.00

II. ELEMENTARY MATHEMATICS.

CLAPHAM (C. B.). Arithmetic for engineers, including simple algebra, mensuration, logarithms, graphs, and the slide rule. London, Chapman and Hall, 1916. 11+436 pp. 5s. 6d.

DICKNETHER (F.). Rechenbuch fur höhere Lehranstalten: 1ter Teil, 5te und 6te Auflage; 2ter Teil. 5te Auflage; 3ter Teil, 4te und 5te Auflage. München, J. Lindauer. 120+142+160 pp. M. 1.70+1.70+1.90.

FENKNER (H.). Mathematisches Uebungsbuch. 2ter Teil (Pensum der Obersekunda und Prima). Ausgabe A (fur Gymnasien); Ausgabe B (für Realgymnasien und Oberrealschulen). Berlin, Salle, 1915. 363+396 pp. M. 3.40+3.60

HUDSON (H. P.). Ruler and compasses. (Modern Mathematical Series.) London, Longmans, 1916. 6+143 pp. 6s.

MAXSON (W. S.). Fundamental number-work method; a logical, practical and systematic method for acquiring thorough knowledge of the foundation processes of fundamental number. Yonkers, N. Y., Self-keyed Number Work Company, 1916. 220 pp. $2.00

MONROE (W. S.). Report of the use of the Courtis standard research tests in arithmetic in twenty-four cities. Emporia, Kansas State Normal School, 1915. 8vo. 94 pp.

PENDLEBURY (C.). Revision papers in arithmetic. London, Bell, 1916. 15+68+18 pp. 1s.

ROBERTSON (J. W.). A shilling arithmetic. London, Bell, 1916. 8+191 pp. 1s.

TALLANT (G. A.). See UMHOLTZ (F. H.).

TALLANT (G. A.) and UMHOLTZ (F. H.). Common school advanced arithmetic. Chicago, Laird and Lee, 1916. 8vo. 6+384 pp. $0.55

UMHOLTZ (F. H.). See TALLANT (G. A.).

UMHOLTZ (F. H.) and TALLANT (G. A.). Common school elementary arithmetic. Chicago, Laird and Lee, 1916. 7+280 pp. $0.35

III. APPLIED MATHEMATICS.

ANNUARIO astronomico pel 1917 publicato dal r. osservatorio di Pino Torinese. Torino, tip. s. Giuseppe degli Artigianelli, 1916. 8vo. 132 pp.

ANNUARIO publicado pelo observatorio nacional do Rio de Janeiro, 1916. Rio de Janeiro, 1916. 6+259 pp.

BARBILLION (L.). Leçons sur le fonctionnement des groupes électrogènes en régime troublé. Paris, Gauthier-Villars, 1916. 2+306 pp. Fr. 11.00

COMINOTTO (E.). Compasso cubico. Piacenza, tip. V. Porta, 1915. 8vo. 10 pp.

DUHEM P.). Le système du monde. Histoire des doctrines cosmologiques de Platon à Copernic. Tomes 2–3. Paris, Hermann, 1914–1915. 522+549 pp.

DUNK (J. L.). Hyperacoustics. Division 1: Simultaneous tonality. London, Dent, 1916. 6+311 pp. 7s. 6d.

HOUSTOUN (R. A.). A treatise on light. London, Longmans, 1915. 11+478 pp. 7s. 6d.

MILLOSEVICH (E.). I pianetini (654) Zelinda e (303) Josephina; Elementi parabolici ed efemeride della cometa 1915 *a* (Mellish); Perturbazioni sulle coordinate per l'azione di Terra a Giove (r. osservatorio astronomico al Collegio Romano). Roma, tip. Unione ed., 1916. 4to. 14 pp.

NAYLER (W. A.). See TUCKEY (C. O.).

ROMEO (A.). Compendio di topografia. Torino, Casanova (Olivero e C.), 1916. 93 pp. con 6 tavole.

ROTIGLIANO (S.). Costruzioni di strade e gallerie. Milano, Hoepli, 1916. 23+808 pp. L. 18.00

RUSSELL (A.). A treatise on the theory of alternating currents. Vol. 2, 2d edition. Cambridge, University Press, 1916. 19+340 pp. 15s.

SEGRETI del giuco del lotto matematicamente studiati e rivelati. Firenze, Emporio poligrafico (stab. tip. Aldino), 1916. 16mo. 14 pp. con prospetto. L. 2.50

TUCKEY (C. O.) and NAYLER (W. A.). Statics: a first course. Oxford, Clarendon Press, 1916. Cr. 8vo. 299 pp. 3s. 6d.

VINASSA DE REGNY (P.). La conquista della forza: discorso letto per la solenne inaugurazione degli studi nella r. Università di Parma il giorno 8 novembre 1915. Parma, tip. E. Pelati, 1916. 8vo. 22 pp.

WEREIDE (T.). Statistical theory of energy and matter. Kristiania, Gyldendalske Boghandel Nordisk Forlag, 1915. 16+170 pp.

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THE TWENTY-THIRD SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-third summer meeting and eighth colloquium of the Society were held at Harvard University during the week September 4–8, 1916. Monday and Tuesday were devoted to the summer meeting proper, two sessions being held on each day for the presentation and discussion of papers. The colloquium opened on Wednesday morning and extended to Friday afternoon. An account of the colloquium appears in this number of the BULLETIN.

The following ninety-nine members attended the summer meeting:

Professor L. D. Ames, Professor R. C. Archibald, Professor G. N. Armstrong, Professor C. S. Atchison, Professor Clara L. Bacon, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor Henry Blumberg, Professor Maxime Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor W. D. Cairns, Professor B. H. Camp, Dr. A. B. Chace, Professor C. W. Cobb, Professor F. N. Cole, Professor L. L. Conant, Professor J. L. Coolidge, Dr. A. R. Crathorne, Professor Louise D. Cummings, Professor C. H. Currier, Professor D. R. Curtiss, Professor E. W. Davis, Dr. C. E. Dimick, Dr. C. R. Dines, Professor Arnold Dresden, Professor Otto Dunkel, Professor W. P. Durfee, Professor W. C. Eells, Professor John Eiesland, Professor L. P. Eisenhart, Professor G. C. Evans, Mr. G. W. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. L. R. Ford, Dr. M. G. Gaba, Professor W. C. Graustein, Dr. G. M. Green, Professor M. W. Haskell, Professor C. N. Haskins, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. M. Hoskins, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor Dunham Jackson, Professor W. W. Johnson, Dr. E. A. T. Kircher, Professor A. E. Landry, Mr. B. B. Libby, Dr. Joseph Lipka, Professor H. P. Manning, Professor J. L. Markley, Professor Helen A. Merrill, Dr. A. L. Miller, Professor H. B. Mitchell, Professor U. G. Mitchell, Professor C. L. E. Moore, Professor C. N. Moore, Professor G. D. Olds, Dr. G. A. Pfeiffer, Professor J. M. Poor, Professor L. H. Rice, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Dr. A. R. Schweitzer, Dr. Caroline E.

Seely, Dr. H. M. Sheffer, Dr. L. L. Silverman, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor R. B. Stone, Professor E. B. Stouffer, Professor H. W. Tyler, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor Roxana H. Vivian, Professor C. A. Waldo, Professor A. G. Webster, Dr. Mary E. Wells, Dr. C. E. Wilder, Professor F. B. Williams, Professor F. N. Willson, Dr. Euphemia R. Worthington, Mr. W. C. Wright, Professor C. H. Yeaton, Professor J. W. A. Young, Professor J. W. Young, Professor Alexander Ziwet.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-Presidents E. R. Hedrick and Virgil Snyder. The Council announced the election of the following persons to membership in the Society: Mr. Herman Betz, Cornell University; Mr. J. A. Bigbee, High School, Little Rock, Ark.; Mr. Hillel Halperin, Vanderbilt University; Dr. J. R. Kline, University of Pennsylvania; Professor J. J. Luck, University of Virginia; Dr. F. J. McMackin, Dartmouth College. Seven applications for membership in the Society were received.

Through the generosity of Harvard University the Freshman Dormitories and dining room were thrown open for the use of the Society during the meeting. On Monday noon the members of the Society were shown the collection of mathematical models belonging to the University. On Wednesday afternoon a visit was paid to the University Library, and on Wednesday evening to the Observatory. Resolutions were adopted at the meeting expressing the thanks of the Society for the hospitality of the University and its officers.

Fraternal greetings were exchanged by cable with the Scandinavian mathematicians assembled at Stockholm. A vote of congratulation was tendered to the Secretary on his twenty-first year of service in that capacity.

The twenty-fifth anniversary of the broadening out of the Society into a national organization and the founding of the BULLETIN were celebrated at the banquet on Monday evening, at which eighty-four members and friends were present. Brief addresses were made by Professors Fiske, Johnson, Fine, Birkhoff, Hedrick, Webster, Coolidge, and the Secretary.

On Tuesday evening Professor D. E. Smith entertained the Society with an instructive account of "The relation of the history of economics to the history of arithmetic problems."

The following papers were read at this meeting:

(1) Professor J. C. FIELDS: "Direct derivation of the complementary theorem."

(2) Dr. C. A. FISCHER: "Note on the order of continuity of functions of lines."

(3) Dr. OLIVE C. HAZLETT: "On the theory of associative division algebras."

(4) Professor W. C. EELLS: "A statistical study of eminent mathematicians."

(5) Professor J. L. COOLIDGE: "The characteristic numbers of real algebraic plane curves."

(6) Dr. R. W. BURGESS: "The comparison of a certain case of the elastic curve with its approximation."

(7) Professor G. A. MILLER: "Orders of operators of congruence groups modulo $2^r 3^s$."

(8) Professor JOHN EIESLAND: "Sphere geometry (third paper)."

(9) Dr. A. J. KEMPNER: "Generalization of a theorem on transcendental numbers."

(10) Professor C. N. MOORE: "On the developments in Bessel's functions."

(11) Professor ARNOLD DRESDEN: "Supplementary note on the second derivatives of an extremal integral."

(12) Professor L. E. DICKSON: "An extension of the theory of numbers by means of correspondences between fields."

(13) Professor A. G. WEBSTER: "On a theory of acoustic horns."

(14) Professor E. H. MOORE: "On properly positive Hermitian matrices."

(15) Professor L. P. EISENHART: "Deformations of transformations of Ribaucour."

(16) Professors F. R. SHARPE and VIRGIL SNYDER: "On (2, 2) point correspondence between two planes."

(17) Professor F. H. SAFFORD: "Surfaces of revolution in the theory of Lamé's products."

(18) Professor DUNHAM JACKSON: "Note on the parametric representation of an arbitrary continuous curve."

(19) Professor DUNHAM JACKSON: "Note on representations of the partial sum of a Fourier series."

(20) Professor L. H. RICE: "Determinants of many dimensions."

(21) Dr. L. R. FORD: "Regular continued fractions."

(22) Professor M. W. HASKELL: "The eliminant of a system of forms."

(23) Professor E. V. HUNTINGTON: "A simple substitute for Duhamel's theorem."

(24) Professor E. V. HUNTINGTON and Dr. J. R. KLINE: "Sets of independent postulates for betweenness."

(25) Professor G. D. BIRKHOFF: "Dynamical systems with two degrees of freedom (second paper)."

(26) Professor E. B. VAN VLECK: "Non-loxodromic substitutions in n variables."

(27) Dr. L. I. HEWES: "Nomograms of adjustment."

(28) Mr. H. C. M. MORSE: "A theorem on the linear dependence of analytic functions of a single variable."

(29) Dr. G. A. PFEIFFER: "Note on the linear dependence of analytic functions."

(30) Dr. G. M. GREEN: "On the linear dependence of functions of one variable."

(31) Professor C. L. BOUTON: "Iteration and group theory."

(32) Dr. G. M. GREEN: "On the general theory of surfaces."

(33) Dr. W. V. GARRETSON: "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution."

(34) Dr. CAROLINE E. SEELY: "On series of biorthogonal functions."

(35) Professor A. B. FRIZELL: "Lemma for a new method of generating alephs."

(36) Professor JOHN EIESLAND: "Transformation theory of the flat complex and its associated line complex."

(37) Dr. A. R. SCHWEITZER: "On a type of quasi-transitive functional equations (second paper)."

(38) Dr. A. R. SCHWEITZER: "A problem in quasi-transitive functional equations."

(39) Dr. A. R. SCHWEITZER: "Some theorems on quasi-transitive functional equations."

(40) Dr. A. R. SCHWEITZER: "On the analogy between functional equations and geometric order relations."

(41) Dr. T. H. GRONWALL: "On the power series for log $(1 + z)$."

(42) Dr. T. H. GRONWALL: "A problem in geometry connected with the analytic continuation of a power series."

(43) Dr. T. H. GRONWALL: "On the convergence of Binet's factorial series for log $\Gamma(z)$ and $\psi(z)$."

(44) Dr. T. H. GRONWALL: "On the zeros of the function $\beta(z)$ associated with the gamma function."

Mr. Morse was introduced by Professor Osgood. The papers of Professor Fields, Dr. Fischer, Professor Miller, Dr. Kempner, Professor Dickson, Professor E. H. Moore, Professor Safford, Dr. Hewes, Dr. Garretson, Dr. Seely, Professor Frizell, Dr. Schweitzer, and Dr. Gronwall were read by title. Abstracts of the papers appear below; the abstracts are numbered to correspond to the titles in the list above.

1. There is nothing in the nature of the complementary theorem which should require for its proof a preliminary determination of a formula for the actual number of the conditions imposed on a rational function of (z, u) of given form by a given set of orders of coincidence corresponding to a specific value of z. The proof here given by Professor Fields dispenses with the aid of such a formula and thus shortens by one step the approach to the theorem. The general complementary theorem then presents itself at a very early stage in the development of the theory of the algebraic functions, the only steps preliminary to its proof, besides those implied in the proof of the existence theorem for the branches of an algebraic function, being those involved in the determination of the reduced form for a rational function of (z, u), those involved in the proof of the integral character relative to the element $z - a$ of the principal coefficient of a rational function of (z, u) which is adjoint for the value $z = a$, and those involved in the proof of what we may call the theorem of principal residues which is derived therefrom and which is contained in the statement: The necessary and sufficient conditions in order that a rational function $H(z, u)$ may have orders of coincidence which are complementary adjoint to a given set of orders of coincidence for the value $z = a$ are obtained on equating to 0 the principal residue relative to $z = a$ in the product of $H(z, u)$ by the general rational function conditioned by the given set of orders of coincidence.

2. Dr. Fischer's paper appears in full in the present number of the BULLETIN.

3. There is a famous theorem to the effect that the only linear associative algebras over the field of real numbers in which division is uniquely possible are the field of real numbers, the field of ordinary complex numbers, and real quaternions, the most recent of the numerous proofs of this being one by Professor Dickson. In his paper the theorem is a special case of a theorem for a certain class of algebras (called Type A) over a general algebraic field F which has a subalgebra S which can be exhibited as an algebra L over a field K with the r^2 units $i^s j^k$ $(s, k = 1, \cdots, r)$, where

$$ji = \theta(i)j, \quad j^r = g(i), \tag{1}$$

where i is an element of S satisfying in K a uniserial abelian equation of degree r, with the roots $\theta(i), \cdots, \theta^r(i) = i$.

Dr. Hazlett's paper considers linear associative division algebras over a general algebraic field F, and finds that a necessary and sufficient condition that such an algebra be of Type A is that θ in (1) be the root of a certain algebraic equation, which we will call the Θ-equation. Then, from some properties of the Θ-equation, this paper proves that, if a linear associative division algebra over an algebraic field be of rank n, it is of order $m \cdot n$ where $m \leqq n$. Of this theorem, Frobenius's theorem is a corollary. It also follows that, if a linear associative division algebra, of rank n, over an algebraic field F, contain a number i which satisfies a uniserial abelian equation of degree n, then any number in the algebra is a polynomial in j with coefficients in $F(i)$, where (1) holds. There are numerous other theorems, some of which are generalizations of well known facts about quaternions.

4. From a representative group of standard histories of mathematics and of general encyclopedias, Professor Eells by the space method selects a group of one hundred most eminent mathematicians of history, discusses the reliability of the selection, and classifies and discusses the men selected according to nationality, period of history in which they flourished, principal branches of mathematics to which they contributed, and other features.

5. A real algebraic plane curve has two sorts of characteristic numbers. First, there are the total or Plücker characteristics, which are invariant for the general plane collineation. Sec-

ond, there are the real characteristics, as the number of real inflections, real double tangents, etc. These are invariant for real collineations. The only known identical equation connecting these latter is the equation of Klein. In Professor Coolidge's paper it is shown that, taking total and real characteristics together, the only universally valid equations are the equations of Plücker and Klein, and others deducible from them.

6. In the usual discussion of the elastic curve in elementary text-books of physics and engineering, a certain term in the differential equation is neglected and the curve obtained as the solution of this approximate equation is said to be a good substitute for the real elastic curve. In the case of a bow or a vertical column this approximate solution is a cosine curve, which is satisfactory if the given constants are the force and the deflection. In the more common problem, however, a bow or column of known length is bent by a known force. Dr. Burgess shows that if a formula for the deflection in terms of the force and the length is deduced from the cosine curve to apply to this case, the result is only about one half the true deflection as found from the real elastic curve.

7. The $\phi(m)$ positive integers which do not exceed the natural number m and are prime to m, when these numbers are combined by multiplication modulo m, constitute a group which is simply isomorphic with the group of isomorphisms of the cyclic group of order m. The principal object of Professor Miller's paper is to show how the order of each of the operators of this group can be directly obtained from the form of the corresponding number by means of general theorems relating to the automorphisms of an abelian group, whenever m is of the form 2^r3^s. Attention is called to the fact that in H. Weber's Lehrbuch der Algebra, second edition, volume 2 (1899), page 65, the exponent to which 5 belongs modulo 2^β is found by a very special method, and that a somewhat similar method is followed by P. Bachmann in the first part of his Niedere Zahlentheorie, 1902, page 331, while the method by means of the theorems noted above exhibits at once the exponent to which each number belongs.

8. Professor Eiesland's memoir is a continuation of a paper on the same subject read before the Society at the

meeting of December 28–29, 1915. An attempt has been made to build up a theory of flat complexes with a view to its application to sphere geometry. It has been shown that with the linear flat complex is associated a line complex which, in some important respects, is analogous to the null-system in ordinary space. The lines of this complex satisfy the system of Pfaffian equations

$$(1)\qquad \begin{aligned} &Qdx_i - Pdy_i + R_i dz = 0, \\ &R_{\frac{1}{2}(n-2)}dy_i - R_i dy_{\frac{1}{2}(n-2)} = 0 \end{aligned} \qquad [i = 1, 2, \cdots, \tfrac{1}{2}(n-2)],$$

where

$$P = A_{\frac{1}{2}n}z + \Sigma A_k x_k - B_0, \quad Q = A_0 z + \Sigma A_k y_k + B_{\frac{1}{2}n},$$
$$R_i = A_{\frac{1}{2}n}y_i - A_0 x_i - B_i,$$

the coefficients A_i, B_i being the constants in the equation of the flat complex

$$(2)\qquad \Sigma A_i \rho_i + \Sigma B_i \sigma_i = 0.$$

All the lines of the line complex which pass through a point form a flat pencil, i. e., they lie in a 2-flat. The relation between point and 2-flat is involutory. Every line is the axis of $\infty^{\frac{1}{2}(n-4)}$ flats of the complex (2).

9. The generalized theorem of Dr. Kempner's paper is as follows:

I. An infinite series

$$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a_n}$$

represents a transcendental number, provided

(1) $\gamma_n = p_n/q_n$ is a real rational number; p_n and q_n are either finite or grow infinite with n in some particular manner; it is for example amply sufficient if $|p_n|$, $|q_n|$ are both $< R^n$, R any fixed positive number;

(2) each denominator a_n is a positive integer and a factor of the next denominator a_{n+1};

$$(3)\qquad \lim_{n=\infty} \frac{a_{n+1}}{a_n^2} \geqq 1$$

if the limit exists; if it does not exist, then

$$\lim_{n=\infty} \inf \frac{a_{n+1}}{a_n^2} \geqq 1.$$

In 1915 (see BULLETIN, March, 1916, page 285) the author presented the theorem

$$\text{I}'. \qquad \sum_{n=1}^{n=\infty} \frac{\gamma_n}{a^{c^n}} \cdot x^n$$

represents a transcendental number, provided

(1′) γ_n restricted as above under (1), but

(2′) only a finite number of the $\gamma_n = 0$,

(3′) a and c any integers $\geqq 2$,

(4′) x any rational number $\neq 0$.

At the April, 1916, meeting at Chicago, Professor H. Blumberg announced that he had succeeded, by a modification of the author's method of proof, in entirely removing the condition (2′). Neither Professor Blumberg's proof nor the author's proof of I′ makes essential use of the assumption that the denominators be of the form a^{c^n}, and analysis of the two proofs shows that it is sufficient to assume that the denominators satisfy (2) and (3) of the generalized theorem.

In Theorem I, nothing is gained by considering the series

$$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a_n} \cdot x^n$$

for rational x, since this is included in I. Theorem I could not be proved before Professor Blumberg's welcome contribution had eliminated (2′).

10. Professor Moore's paper appeared in full in the October BULLETIN.

11. In a paper presented to the Society at its meeting in Chicago, December, 1914, Professor Dresden gave formulas for the second derivatives of the extremal integrals arising in the theories of the integrals

$$\int f(x; y_1, \cdots, y_n; y_1', \cdots, y_n')dx$$

and

$$\int f(y_1, \cdots, y_n; y_1', \cdots, y_n')dt.$$

A certain lack of symmetry in the formulas for the second of these theories made them rather unsuitable for applications and undesirable from a theoretical point of view. By using the "normal" solutions of the system of Jacobi differential

equations recently introduced by Professor Bliss, it has been found possible to remove this lack of symmetry and to simplify the formulas very materially. It is the object of the present note to derive these simplified formulas.

12. Professor Dickson's paper will appear in full in an early number of the BULLETIN.

13. Since no boundary problem for an unclosed non-plane surface has been solved with the exception of Kelvin's spherical bowl, there is as yet no theory of horns of finite length. As such a theory is eminently desirable not only for musical instruments but for reception apparatus, including the phonograph, megaphone, and a great variety of acoustical instruments, Professor Webster has been driven to a very coarse approximate theory, which works surprisingly well in practice. Two assumptions are made, first that the cross-section of the horn is infinitesimal with respect to the wave length, and varies as any given function of the distance from a certain section; second that at the open end Helmholtz's assumption for tubes of infinitesimal cross-section may be used. The first assumption leads to a linear differential equation, and when the section varies as a power of the axial distance this may be integrated by Bessel functions of fractional index. The cases of cylinders, cones, and hyperbolic horns such as are used in the orchestra come under this case. The assumption for the end, even when the diameter is half the wave length instead of an infinitesimal part of it, is found to agree within a few per cent with experimental determinations. Any horn is characterized by four constants, and may be replaced by any other horn whatever having the same constants.

14. The new theory of linear integral equations in general analysis, of which a special instance is Hilbert's theory of limited symmetric bilinear forms in infinitely many variables, under development by Professor Moore, is based on a properly positive Hermitian matrix or numerically valued function ϵ or $\epsilon(s, t)$ of two variables s, t ranging independently over a general class $\mathbf{P}$, viz., a function ϵ of such a nature that (1) $\overline{\epsilon(s, t)} = \epsilon(t, s)$ for every s, t of $\mathbf{P}$; (2) every finite principal minor is positive, i. e., for every set $(p_1, p_2, \cdots, p_n)$ of a

finite number n of distinct elements of $\mathbf{P}$, the determinant of the n^2 functional values $\epsilon(p_i, p_j)(i, j = 1, 2, \cdots, n)$ is positive. For the Hilbert instance, the class $\mathbf{P}$ is the class of positive integers, and the function ϵ is δ, $\delta(s, t) = d_{st} = 1(s = t)$, $0(s \neq t)$.

The present paper of Professor Moore presents various absolute and relative theorems of existence of such matrices ϵ. As an example, if $\mathbf{P}$ is the class of all real valued functions p or $p(x)$ on the interval 01, ϵ or $\epsilon(s, t) = \exp \int_0^1 s(x)t(x)dx$ is such a matrix ϵ. This example is of especial importance in that for the corresponding instance of the general theory we meet for the first time a non-trivial instance of an integration process whose integrand functions are themselves functional operations; this "integration over a function space" is secured in virtue of the fact that the integration process involves in its definition no assumed metrical features of the range of integration.

15. When a system of spheres involves two parameters, their envelope consists, in general, of two sheets, say Σ and Σ_1, and the centers of the spheres lie upon a surface S. A correspondence between Σ and Σ_1 is established by making correspond the points of contact on the same sphere. When the lines of curvature on Σ and Σ_1 correspond, Σ_1 is said to be in the relation of a transformation of Ribaucour with Σ, and vice versa. It is a known property of envelopes of spheres that if the surface of centers S be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before deformation. Professor Eisenhart determines the transformations of Ribaucour which in deformations of the surface of centers S become transformations of Ribaucour. The surfaces Σ in this case have the same spherical representation of their lines of curvature as isothermic surfaces. The paper will be published in the October number of the *Transactions*.

16. Marletta (*Palermo Rendiconti*, volume 17 (1903)) has discussed the (2, 2) correspondence between the points of two planes in which a line is transformed into a conic or a cubic. In the paper of Professors Sharpe and Snyder the two images

of a point in one plane are the intersections of a corresponding line and conic in the other. The image of a line is a quintic of genus 2. The correspondences are classified according to the number of basis points the nets of conics and the quadratic systems of lines may have.

The results obtained by Marletta and those known earlier all appear as particular cases.

17. In the investigation of surfaces of revolution in the theory of Lamé's products it is necessary to find a function of two variables such that a certain known function of the original one shall be the sum of two functions, one of the first variable and the other of the second. After the original function has been determined, the resolution of the known function into a sum as stated has required separate treatment for each case considered. In this paper Professor Safford has obtained the general expression for the component parts of the sum in terms of the original function. The meridian curves are in general Weierstrass $\wp$-function curves. The paper will be sent to the *Archiv der Mathematik und Physik,* in which several related papers have already appeared.

18. A continuous curve may be defined by a pair of equations

$$x = f(t), \qquad y = \varphi(t),$$

where the functions f and φ are defined and continuous throughout some interval, and are not both constant throughout that interval. For some discussions it is convenient to suppose further that they are not simultaneously constant throughout any sub-interval of their interval of definition. The question arises, whether this assumption implies a restriction on the curve itself, that is, on the set of points (x, y) given by the equations, or whether it is a restriction merely on the particular parametric representation employed. It is obvious that the latter is the case, if the number of intervals of constancy is finite.

It was proved by Fréchet (*Rendiconti Palermo,* volume 22) that this remains true, even when there are infinitely many of the intervals in question. Professor Jackson gives a different proof of the same theorem.

19. In the general theory of the convergence of expansions in series associated with ordinary linear differential equations,

as developed by Birkhoff (*Transactions*, 1908) and others, the sum of the first n terms of the series is represented by a contour integral in the plane of the complex parameter. The contour integral, in the fundamental particular case of Fourier's series, must of course be identical in value with the ordinary real formula for the partial sum of the series. In his second note, Professor Jackson enumerates the steps in the process of verifying this identity by direct evaluation of the contour integral, without going back to the individual terms which make up the sum. The Fourier's series can be derived either from a system of the first order or from one of the second order, and the work is carried through for both cases.

20. An extended definition of a determinant is given by Professor Rice, which applies to determinants of more than three dimensions and enables us to remove the restriction in Cayley's law of multiplication and to set up a new case in Scott's law of multiplication. New formulas are obtained for the known process of decomposition of a determinant into determinants of fewer dimensions, and a new process called crossed decomposition is described. Fresh light is thrown upon the function known as a "determinant-permanent," a limitation hitherto thought necessary being done away. Finally a generalization to p dimensions is made of Metzler's theorem in two dimensions concerning a determinant each of whose elements is the product of k factors.

21. A continued fraction

$$b_0 \pm \frac{a_1}{b_1 \pm} \frac{a_2}{b_2 \pm} \cdots \frac{a_n}{b_n \pm} \cdots$$

is called regular if $a_i = \pm 1$ or $\pm \sqrt{-1}$, and

$$b_i = m_i + n_i \sqrt{-1},$$

where m_i and n_i are real integers. In Dr. Ford's paper the classic pentahedral division of half-space corresponding to the group of Picard is employed to give a geometrical interpretation of the convergents of the continued fraction.

It is shown how any such continued fraction can be determined by means of a curve lying in the half-space. Explicit

formulas are given for setting up the continued fraction corresponding to a given curve. By a study of the generating curve general theorems are established relative to (1) the value of the convergents as approximations to the sum of the continued fraction, and (2) the periodicity of the continued fraction.

22. Professor Haskell gives a method of constructing the eliminant of a system of forms which leads to a direct proof of the well-known theorem that the number of solutions of a system of simultaneous equations is equal to the product of their degrees. Beginning with a system of linear forms, he uses the theory of symmetric functions to construct a series of eliminants, in which the linear forms are, one after the other, replaced by forms of higher degree, and thus arrives at the contravariant which factors into the linear factors corresponding to the various solutions of the system of forms of any degree, and without any extraneous factor.

23. Professor Huntington's note contains the following substitute for Duhamel's theorem, which may seem simpler than the substitutes already published in the *Annals of Mathematics* by Osgood, R. L. Moore, and Bliss: Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in a way that suggests dividing the interval into n small parts, or elements, and regarding P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $f(x)$, $F(x)$, $\cdots$, can be found such that no matter what the value of n, the contribution of a typical element, from $x = x$ to $x = x + \Delta x$, can be expressed "approximately" in the form $[f(x)\cdot F(x)\cdots]\Delta x$—that is, so that the exact value of the contribution lies between $[f_1\cdot F_1\cdots]\Delta x$ and $[f_2\cdot F_2\cdots]\Delta x$, where f_1, F_1, $\cdots$ are the least, and f_2, F_2, $\cdots$ the greatest values of $f(x)$, $F(x)$, $\cdots$ in the interval in question. Then the required quantity P can be computed as a definite integral: $P = \int_a^b [f(x)\cdot F(x)\cdots]\, dx$, provided only that the functions involved are continuous from $x = a$ to $x = b$.

24. The paper of Professor Huntington and Dr. Kline starts with the following basic list of twelve postulates for be-

tweenness: (*A*) If *AXB* then *BXA*. (*B*) If *A*, *B*, *C* are distinct, then at least one of the relations *BAC*, *CAB*, *ABC*, *CBA*, *ACB*, *BCA* will be true. (*C*) *AXY* and *AYX* cannot both be true. (*D*) If *ABC* is true, then the elements *A*, *B*, *C* are distinct. Further, if *A*, *B*, *X* and *Y* are distinct, then: (1) If *XAB* and *ABY* then *XAY*; (2) If *XAB* and *AYB* then *XAY*; (3) If *XAB* and *AYB* then *XYB*; (4) If *AXB* and *AYB* then *AXY* or *AYX*; (5) If *AXB* and *AYB* then *AXY* or *YXB*; (6) If *XAB* and *YAB* then *XYB* or *YXB*; (7) If *XAB* and *YAB* then *XYA* or *YXA*; (8) If *XAB* and *YAB* then *XYA* or *YXB*. The question then is: Given any subset *S* of the twelve postulates of this basic list, and any postulate *P* of the list, not belonging to *S*; is *P* deducible from *S*? An exhaustive answer to this question is given in 71 theorems of deducibility and 68 theorems of non-deducibility, these latter being established by the aid of 44 examples of pseudo-betweenness. It is found that eleven different sets of independent postulates may be selected from the basic list, as follows: 1, 2; 1, 5; 1, 6; 1, 7; 1, 8; 2, 4; 2, 5; 3, 5; 3, 4, 6; 3, 4, 7; 3, 4, 8, with the addition of postulates *A*, *B*, *C*, *D* in each of these sets. The paper will be offered to the *Transactions*.

25. Professor Birkhoff proves two theorems concerning the invariant points of a surface of any genus p under a one-to-one transformation into itself. The second of these theorems, in which it is assumed that the transformation leaves an area integral invariant, is based upon a modification of Poincaré's last geometric theorem. The results admit of application to dynamical systems with two degrees of freedom.

26. At the last meeting of the Society in Chicago Professor Van Vleck presented a paper on the "Composition of non-loxodromic linear substitutions, $z' = (az+b)/(cz+d)$." When expressed in unimodular form

$$z_1' = az_1 + bz_2, \quad z_2' = cz_1 + dz_2$$

with unit determinant, the non-loxodromic substitution has a real characteristic equation. Professor Van Vleck defined correspondingly a non-loxodromic linear substitution in n homogeneous variables to be one which, expressed in unimodular form, has a real characteristic equation. He then

announced his first results in extension of the preceding paper, for proof of which new methods were found necessary.

27. D'Ocagne has shown that an equation of the form

$$(1) \qquad \begin{vmatrix} f_{12} & g_{12} & h_{12} \\ f_{34} & g_{34} & h_{34} \\ f_{56} & g_{56} & h_{56} \end{vmatrix} = 0,$$

where f_{ij}, g_{ij}, h_{ij} are functions of the same two variables, may be represented by a nomogram with at most three curve nets. It is proposed by Dr. Hewes to generalize the form of equation (1) by the use of the equation

$$(2) \qquad \begin{vmatrix} f_{ij} & g_{ij} & h_{ij} \\ f_{kl} & g_{kl} & h_{kl} \\ f_{mn} & g_{mn} & h_{mn} \end{vmatrix} = 0,$$

where the subscripts may take on the values 0, 1, 2, $\cdots$, 6. There result nomograms requiring rotation of the index and of wide application.

28. Mr. Morse proves the following theorem, which is due to Professor Osgood:

Let $f_1(t), f_2(t), \cdots, f_p(t)$ be p functions of the complex variable t, analytic in a region S of the t-plane. Then a necessary and sufficient condition for the linear dependence of these functions is the identical vanishing of a p-rowed determinant whose ith row ($i = 1, 2, \cdots, p$) is $f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_1-1)}(t_1), \cdots, f_i'(t_1), f_i(t_1), f_i^{(\lambda_2)}(t_2), f_i^{(\lambda_2-1)}(t_2), \cdots, f_i'(t_2), f_i(t_2), \cdots, \cdots, f_i^{(\lambda_\mu)}(t_\mu), f_i^{(\lambda_\mu-1)}(t_\mu), \cdots, f_i'(t_\mu), f_i(t_\mu)$, where $t_1, t_2, \cdots, t_\mu$ are independent variables and the upper indices denote differentiation.

The theorem includes the ordinary wronskian theorem for analytic functions as a special case, and the method of proof affords a new demonstration for the latter theorem.

29. Dr. Pfeiffer proves the theorem given by Mr. Morse by showing that successive differentiation of the determinant in question, and subsequent equating of all the independent variables, reduces the determinant to the wronskian of the given functions.

30. In his first paper Dr. Green establishes a condition under which the vanishing of the determinant of the two preceding papers is sufficient for the linear dependence of the functions involved, these functions not being required to be analytic. The existence of only those derivatives which appear in the said determinant is required, the only restriction being the non-vanishing of a certain first minor in the determinant. The proof is analogous to the one given by Bôcher* for the corresponding wronskian theorem.

31. Professor Bouton's note gives a discussion of a method of finding all the continuous one-parameter groups which contain a given individual point transformation in n variables. This given transformation is taken in the form $x_i' = x_i + \psi_i(x)$ where the ψ_i are convergent power series beginning with terms of the second degree. In the course of the work the result of iterating the given transformation k times is derived.

32. In extending theorems of metric differential geometry to projective, it is necessary to substitute for the congruence of normals to a surface a congruence which is likewise uniquely determined by the surface, but projectively. Wilczynski's directrix congruence of the second kind might be made to serve the purpose; but this congruence has not the peculiar projective property of the normal congruence, whose developables intersect the surface in a conjugate net. In his second paper, Dr. Green defines and characterizes geometrically a congruence which has this desired property. In a previous paper† he defined a relation R, by means of which to every line through a point P of the surface is made to correspond a unique line lying in the tangent plane, and vice versa—the correspondence depending upon the parametric net. Let the surface be referred to its asymptotic curves, and let l be a line in the tangent plane, l' the corresponding line through the point P. Project the asymptotic curves on the tangent plane from any point on l', and construct the conics C_1 and C_2 which osculate these projections at the point P. Then there exists but one pair of lines l, l' such that the intersec-

* "Certain cases in which the vanishing of the wronskian is a sufficient condition for linear dependence," *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 139–149, Theorem II.

† "On rectilinear congruences and nets of curves on a surface." Abstract in the BULLETIN, vol. 22 (1916), p. 274.

tions of l with the asymptotic tangents are the double points of the involution determined by the two pairs of points in which l cuts the conics C_1 and C_2. Let L_1 and L_2 be the intersections of the asymptotic tangents with the line l thus uniquely determined, and denote by D_1 and D_2 the intersections of the asymptotic tangents with the directrix of the first kind. Calling P_1 the harmonic conjugate of D_1 with respect to P and L_1, and P_2 the harmonic conjugate of D_2 with respect to P and L_2, one thus obtains a line m joining the points P_1 and P_2 and lying in the tangent plane to the surface. Corresponding to the line m, in the relation R, there exists a line m' passing through the point P. For every point P of the surface exists such a line m', and the congruence of these lines has the property that its developables cut the surface in a conjugate net.

Among other results arrived at may be mentioned a geometric characterization of the tetrahedron of reference which gives rise to a certain canonical development of the non-homogeneous coordinates of a point of the surface. This development was given by Darboux without any geometric characterization. The points L_1 and L_2 of the preceding investigation are two of the vertices of the tetrahedron, and the line m' is an edge.

33. Dr. Garretson's paper treats of the irregular integrals of the non-homogeneous linear equation of any order, where the roots of the characteristic equation are distinct. The irregular point is taken as the point at infinity. Two theorems given by Dini for the homogeneous equation have been generalized so as to apply to the non-homogeneous equation and combined in one theorem. To this end certain ideas given by Horn have been employed. Use is also made of the solutions obtained by Professor C. E. Love for the homogeneous equation. The paper leads to the asymptotic development, in the Poincaré sense, of a particular solution of the non-homogeneous equation.

34. A necessary and sufficient condition for the essentially uniform convergence of series of orthogonal functions in two variables

$$\sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(y)}{\lambda_i}$$

has been given by Lauricella. Miss Seely considers the corresponding problem for series of the form

$$\sum_{i=1}^{\infty} \frac{\varphi_i(x)\psi_i(y)}{\lambda_i},$$

where the φ's and ψ's are a biorthogonal system of functions, and obtains conditions of which Lauricella's theorem is a special case.

35. In the November BULLETIN* Professor Frizell used a simple special case of the following lemma. Given a triply infinite partition $u_{\lambda\mu\nu}$ of an infinite set of symbols v_ι selected from an ω-series a_ι $(i, \iota, \lambda, \mu, \nu = 1, 2, \cdots)$ and a non-enumerable well-ordered set P of different permutations among the members of an ω-series; if now the elements $u_{\lambda\mu\nu}$ are exhibited in a rectangular array of infinite series, wherein λ, μ denote the row and column respectively:

$$\begin{array}{llll}
u_{111}, u_{112}, \cdots; & u_{121}, u_{122}, \cdots; & u_{131}, u_{132}, \cdots; & \cdots \\
u_{211}, u_{212}, \cdots; & u_{221}, u_{222}, \cdots; & u_{231}, u_{232}, \cdots; & \cdots \\
u_{311}, u_{312}, \cdots; & u_{321}, u_{322}, \cdots; & u_{331}, u_{332}, \cdots; & \cdots \\
u_{411}, u_{412}, \cdots; & u_{421}, u_{422}, \cdots; & u_{431}, u_{432}, \cdots; & \cdots \\
\cdots & \cdots & \cdots &
\end{array}$$

then the permutations P operating on the indices ν (each pair λ, μ being held fast) produce new matrices $\mathfrak{M}$ yielding permutations $\mathfrak{P}$ which can not be put into one-to-one correspondence with the set P. The present note is devoted to the proof of the general proposition.

36. To the conformal group in S_{n-1} (n even) corresponds in $\bar{S}_{n-1}$ a group of contact transformations $G_{\frac{1}{2}(n+1)(n+2)}$ which transforms the ∞^n flats

$$(1) \qquad \begin{gathered} x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d, \\ az = \Sigma c_i x_i + ad - \Sigma c_i b_i \end{gathered}$$

inter se. The group also preserves asymptotic lines on a surface. An important subgroup of this group is considered by Professor Eiesland, viz.:

* Vol. 22, No. 2, p. 73.

1. The group of the complex $a + d = 0$ which carries null-flats into null-flats, with which is closely connected
2. The group of the associated line complex, and also
3. The group of parameters which shows how the flats (1) are transformed when the surface elements of $\bar{S}_{n-1}$ are subjected to the transformations of $G_{\frac{1}{2}(n+1)(n+2)}$. This group is the group of the quadric surface

$$e = ad - \Sigma b_i c_i$$

so that the geometry of flats is equivalent to a geometry on a quadric surface in a space of $n + 1$ dimensions.

The following theorem is also proved:

It is always possible by means of a linear transformation to reduce the system of Pfaffian equations

$$Qdx_i - Pdy_i + R_i dz = 0, \quad R_{\frac{1}{2}(n-2)}dy_i - R_i dy_{\frac{1}{2}(n-2)} = 0$$

to any one of the following forms:

(1) $dx_i - zdy_i + y_i dz = 0, \quad y_{\frac{1}{2}(n-2)}dy_i - y_i dy_{\frac{1}{2}(n-2)} = 0,$

or

(2) $zdx_i - x_i dz = 0, \quad x_i dy_{\frac{1}{2}(n-2)} - x_{\frac{1}{2}(n-2)}dy_i = 0,$

or

(3) $dx_i = 0.$

37. In a former paper Dr. Schweitzer considered quasi-transitive functional equations of the type

(1) $$f(u_1, u_2, \cdots, u_{n+1}) = f(y_1, y_2, \cdots, y_{n+1}),$$

where

$$u_i = f(t_1, t_2, \cdots, t_n, x_i), \quad y_i = \sum_{k=1}^{n+1} m_{ik}x_k,$$

and it was shown that if

$$f(x_1, x_2, \cdots, x_{n+1}) = \sum_{k=1}^{n+1} \alpha_k x_k$$

then

$$\sum_{k=1}^{n+1} \alpha_k = 0$$

and $\alpha_2, \alpha_3, \cdots, \alpha_{n+1}$ respectively satisfy n algebraic equations of the nth degree with coefficients in the field of

$$M_{st} = m_{st} - m_{1t} \quad (s, t = 2, 3, \cdots, n + 1).$$

Each of the roots α is, moreover, a rational function of α_{n+1} with regard to M_{st} $(s, t \neq 1)$.

In the present paper, the problem of deriving conditions on the M_{st} such that the n equations of the nth degree possess the same roots $\alpha_2, \alpha_3, \cdots, \alpha_{n+1}$ is discussed. In this case, the resulting single equation, if irreducible in the field of certain of the M's in which the coefficients of the equation lie, is regular with reference to this field. Conversely, if the n relations (derived from the equation (1)) in terms of $\alpha_2, \cdots, \alpha_{n+1}$ and rational with regard to M_{st}, are subjected to the substitutions of a regular group on the symbols $\alpha_2, \cdots, \alpha_{n+1}$, then from the resulting n sets of n relations it follows that $\alpha_2, \cdots, \alpha_{n+1}$ are roots of the same equation. This consideration leads to sufficient conditions for the desired properties of the $\alpha_2, \cdots, \alpha_{n+1}$. For example, when $n + 1 = 3, 4, 5$ the author obtains four theorems corresponding to the four regular groups on two, three and four symbols:

I. If

$$f(u_1, u_2, u_3) = f(-bx_2, -bx_3, -ax_1 + (a - b)x_3)$$

then α_2, α_3 are roots of the quadratic $\xi^2 - a\xi + ab = 0$.

II. If

$$f(u_1, u_2, u_3, u_4) = f(y_1, y_2, y_3, y_4),$$
$$f(v_1, v_3, v_4, v_2) = f(y_1, y_3, y_4, y_2),$$

where

$$v_i = f(t_1, t_2, x_i, t_3)$$

then $\alpha_2, \alpha_3, \alpha_4$ are roots of the cubic $\xi^3 - a\xi^2 + ab\xi - ac = 0$ provided

$$\sum_{t=1}^{4} M_{st} = 0 \ (s = 2, 3, 4), \quad M_{21} = 0, \quad M_{31} = 0,$$

$$M_{22} = M_{43}, \quad M_{33} = M_{42}, \quad M_{23} = M_{32},$$
$$M_{22} + M_{33} + M_{44} = a, \quad M_{22} + M_{33} + M_{23} = b,$$
$$M_{22}M_{33} - M_{23}^2 = c, \quad M_{23}M_{44} = c - (M_{22}^2 + M_{33}^2).$$

III. If

$$f(u_1, u_2, u_3, u_4, u_5) = f(y_1, y_2, y_3, y_4, y_5),$$
$$f(v_1, v_3, v_2, v_5, v_4) = f(y_1, y_3, y_2, y_5, y_4),$$
$$f(w_1, w_4, w_5, w_2, w_3) = f(y_1, y_4, y_5, y_2, y_3),$$

where

$$v_i = f(t_1, t_2, t_3, x_i, t_4), \quad w_i = f(t_1, t_2, x_i, t_3, t_4)$$

then $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ are roots of the biquadratic $\xi^4 - a\xi^3 + ab\xi^2 - ac\xi + ad = 0$ provided

$$\sum_{t=1}^{5} M_{st} = 0 \ (s = 2, 3, 4, 5), \quad M_{21} = 0, \quad M_{31} = 0,$$

$$M_{41} = 0, \quad M_{22} = M_{52}, \quad M_{33} = M_{53}, \quad M_{44} = M_{54},$$

$$M_{32} = M_{42}, \quad M_{23} = M_{43}, \quad M_{24} = M_{34},$$

$$M_{22} + M_{33} + M_{44} + M_{55} = a$$

and M_{55}, b, c, d are certain rational functions of $M_{22}, M_{33}, M_{44}, M_{23}, M_{32}, M_{24}$.

IV. This theorem is analogous to theorem III. Presumably one obtains for $n + 1 > 5$ theorems on algebraic equations corresponding to every regular substitution group on n symbols.

38. Dr. Schweitzer proposes the following problem: Let

$$(1) \qquad f(u_1, u_2, \cdots, u_{n+1}) = \phi(x_1, x_2, \cdots, x_{n+1}),$$

where

$$u_i = f(t_1, t_2, \cdots, t_n, x_i) \quad (i = 1, 2, \cdots, n + 1);$$

to determine the function f under the following conditions: If

$$f(x_1, x_2, \cdots, x_{n+1}) = f_0(x_1, x_2, \cdots, x_{n+1})$$

is a solution of (1) so also is

$$f(x_1, x_2, \cdots, x_{n+1}) = f_0(x_1, \cdots, x_k, x_{i_{k+1}}, \cdots, x_{i_{k+j}}, x_{k+j+1}, \cdots, x_{n+1}),$$

where $i_{k+1}, \cdots, i_{k+j}$ each range over $k + 1, k + 2, \cdots, k + j$ and are distinct and

$$\begin{pmatrix} (k+1), & \cdots, & (k+j) \\ i_{k+1}, & \cdots, & i_{k+j} \end{pmatrix}$$

is any given substitution of any given group G on the symbols $x_{k+1}, \cdots, x_{k+j}$. Two important special cases of this problem are (I) the determination of a class of solutions which are permuted among themselves when subjected to the substitutions of the group G when the latter is supposed, e. g., regular; (II) the determination of a solution of (1) which is invariant under the substitutions of G. The linear func-

tions discussed in the preceding paper of the author yield solutions of problems of type (I), whereas corresponding to (II) the following theorem is proved: Given any substitution group G

$$G: \qquad \left\{ \begin{matrix} X_2, & \cdots, & X_n \\ X_{i_2}, & \cdots, & X_{i_n} \end{matrix} \right\} (X_{n+1})$$

there exists a group of quasi-transitive functional equations, namely,

$$E \left\{ \begin{matrix} 1, & 2, & 3, & \cdots, & n, & n+1 \\ 1, & i_2, & i_3, & \cdots, & i_n, & n+1 \end{matrix} \right\}$$

simply isomorphic with G and which have as a common particular solution a function which is formally invariant under the substitutions of the group G.

39. Dr. Schweitzer proves the following theorems:

I. If

$$f\{z, f(x, y)\} = f\{y, f(x, z)\}, \quad \phi\{f(x, y), x\} = \psi(y),$$
$$f\{\phi(x, y), x\} = \psi(y), \quad f\{\psi(x), \psi(y)\} = f(y, x),$$

then there exists $\chi(x)$ such that

$$\chi f(x, y) = \chi(x) - \chi(y), \quad \chi\phi(x, y) = \chi(x) - \chi(y) + c,$$
$$\chi\psi(x) = c - \chi(x)$$

where c is an arbitrary constant. Therefore

$$\phi\{\phi(x, y), \phi(x, z)\} = \phi(z, y)$$

and

$$f\{f(x, y), f(x, z)\} = f(z, y).$$

II. If

$$f\{u_1, u_2, \cdots, u_{n+1}\} = f\{\theta(x_1), \cdots, \theta(x_{n+1})\}$$

where

$$u_i = f\{t_1, t_2, \cdots, t_n, x_i\} \quad (i = 1, 2, \cdots, n+1)$$

then

$$\psi f(x_1, \cdots, x_{n+1}) = \Theta\{\psi(x_1) - \psi(x_2), \cdots, \psi(x_1) - \psi(x_n)\}$$
$$+ k(\psi(x_i) - \psi(x_{n+1})) + c_1,$$
$$\psi\theta(x) = -k\psi(x) + c_2$$

where k, c_1, and c_2 are arbitrary constants and Θ is an "arbitrary" function of $(n-1)$ variables.

III. If

$$f\{u_1, u_2, \cdots, u_{n+1}\}$$
$$= f\{\alpha_1(x_1), \alpha_2(x_3), \alpha_3(x_4), \cdots, \alpha_n(x_{n+1}), \alpha_{n+1}(x_2)\}$$

then

$$\psi f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} a_i \psi(x_i) + l,$$

$$\psi\alpha_i(x) = m_i\psi(x) + p_i \quad (i = 1, 2, \cdots, n+1),$$

where

$$\sum_{i=1}^{n+1} a_i p_i = 0, \quad \sum_{i=1}^{n+1} a_i = 0, \quad a_2 = m_{n+1}, \quad a_{n+1} = m_1,$$

$$a_{j+1} = \frac{m_2 m_3 \cdots m_j}{m_1^{j-1}} \cdot m_{n+1} \quad (j = 2, 3, \cdots, n-1)$$

and $m_1^n = m_2 m_3 \cdots m_n m_{n+1}$. The m's and p's are constants arbitrary apart from the conditions indicated.

40. The object of Dr. Schweitzer's fourth note is to point out an analogy that exists between geometric order relations in the foundations of geometry and functional equations. This analogy consists in interpreting functionally abstractions of geometric order relations; in other words, it is shown that certain geometric order relations and functional equations give rise to the same general abstract relations resembling the "associations" of Grassmann and Pietzker. The geometric order relations are based* mainly on the author's "right-handedness" relation $\alpha R_n \beta_1, \beta_2, \cdots, \beta_n$ and "interior" relation $\alpha I_n \beta_1 \beta_2 \cdots \beta_{n+1}$ and the functional equations arrived at belong to categories previously defined by the author, especially the "quasi-transitive" functional equations. The general logical theory of relations relevant to the preceding position is that of Veronese.

41. Dr. Gronwall proves by actual analytic continuation of the power series $\log(1+z) = z - z^2/2 + z^3/3 - \cdots$ the familiar facts that $\log(1+z)$ has its only finite singular point at $z = -1$, and that the continuation along a closed path winding once in the positive direction around $z = -1$ has the affect of increasing $\log(1+z)$ by $2\pi i$.

* Cf. *Amer. Jour. of Mathematics*, 1909, pp. 365–410.

42. Dr. Gronwall solves the following problem: given a power series in z defining an analytic function with the sole singularity $z = -1$ at finite distance, what is the minimum number m of intermediate points $a_1, a_2, \cdots, a_m$ needed to effect the analytic continuation of the given power series to a given point z_0? Writing $z = -1 + re^{\theta i}$, $-\pi \leqq \theta \leqq \pi$, it is found that the set of points z_0, for which no more than m intermediate points are necessary, is formed by the interior of the curve

$$r = \left(2 \cos \frac{\theta}{m+1}\right)^{m+1}, \quad (-\pi \leqq \theta \leqq \pi).$$

The totality of possible positions of the last intermediate point a_m for a given z_0 is also determined.

43. The four series in question are known to converge absolutely and uniformly for $R(z) \geqq \epsilon > 0$, but diverge for $R(z) < 0$ ($R(z)$ = real part of z, and ϵ arbitrarily small). Dr. Gronwall investigates the convergence on the boundary line $R(z) = 0$, and finds that all four series converge uniformly for $R(z) \geqq 0$, $|z| \geqq \epsilon$; two of the series are absolutely convergent in the same region, while the remaining two do not converge absolutely for any purely imaginary value of z.

44. In his Handbuch der Gammafunktion, Nielsen shows that the function

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z}{2}\right) + \psi\left(\frac{1-z}{2}\right)\right],$$

where $\psi(z) = d \log \Gamma(z)/dz$, has no real zeros, and raises the question whether any complex zeros exist. In the present paper, Dr. Gronwall establishes the existence of an infinity of such zeros, and gives their asymptotic expressions.

F. N. Cole,
Secretary.

THE CAMBRIDGE COLLOQUIUM.

The eighth colloquium of the American Mathematical Society was held in connection with its twenty-third summer meeting at Harvard University, Cambridge, Massachusetts. At the April, 1915, meeting of the Council the invitation of

the Division of Mathematics of Harvard University to hold the summer meeting and colloquium at Harvard was accepted, and a committee of arrangements appointed, consisting of Professors Osgood, Bôcher, E. H. Moore, P. F. Smith, and the Secretary. The courses of lectures were announced in the preliminary circular of April, 1916, and printed syllabi were distributed at the meeting. The colloquium opened Wednesday morning, September 6, and continued until Friday afternoon; three lectures were delivered on each of the first two days, and four on Friday. A number of books of reference were placed on the reserved shelves of the university library for the use of those attending the colloquium, and ample opportunity for conferences was provided in Smith and Standish Halls throughout the week. The following sixty-nine persons were in attendance, a number considerably exceeding that of any previous colloquium.

Professor L. D. Ames, Professor R. C. Archibald, Professor Clara L. Bacon, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor H. Blumberg, Professor M. Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. R. W. Burgess, Dr. A. B. Chase, Professor C. W. Cobb, Professor F. N. Cole, Professor J. L. Coolidge, Dr. A. R. Crathorne, Professor Louise D. Cummings, Professor C. H. Currier, Miss A. M. Curtis, Dr. C. E. Dimick, Professor A. Dresden, Professor Otto Dunkel, Professor J. Eiesland, Professor G. C. Evans, Mr. G. W. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. L. R. Ford, Dr. M. G. Gabà, Professor W. C. Graustein, Dr. G. M. Green, Professor M. W. Haskell, Professor C. N. Haskins, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. M. T. Hu, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor D. Jackson, Mr. R. Keffer, Dr. E. A. T. Kircher, Professor A. E. Landry, Mr. F. W. Loomis, Professor J. L. Markley, Professor Helen A. Merrill, Dr. A. L. Miller, Professor H. B. Mitchell, Professor C. N. Moore, Dr. F. D. Murnaghan, Dr. F. H. Murray, Dr. G. A. Pfeiffer, Dr. T. A. Pierce, Professor R. G. D. Richardson, Dr. A. R. Schweitzer, Dr. Caroline E. Seely, Dr. L. L. Silverman, Professor Clara E. Smith, Professor V. Snyder, Professor R. P. Stephens, Professor R. B. Stone, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor A. G. Webster, Dr. Mary E. Wells, Dr. C. E. Wilder, Dr. Euphemia R. Worthington, Professor J. W. Young, Professor Alexander Ziwet.

Two courses of five lectures each were given:

I. Professor G. C. EVANS: "Topics from the theory and applications of functionals, including integral equations."

II. Professor O. VEBLEN: "Analysis situs."

Abstracts of the lectures follow below. The lectures will soon be published in full as Volume V of the Colloquium Series.

I.

LECTURE I. FUNCTIONALS OF CURVES AND SURFACES IN THREE DIMENSIONS; VARIATIONAL EQUATIONS.

1. Additive and non-additive functionals of curves in two and three dimensions; existence of a derivative; relation to functions of point sets.

2. The vector of a functional.

3. Variation of a functional:

$$\delta\Phi[c] = \int_c \Phi'[c|s]\delta n(s)ds.$$

$$\delta\Phi'[c|s] = \int_c \Phi''[c|s_1, s]\delta n(s_1)ds_1 + a_0(s)\delta n(s) + \sum_1^n a_i(s)\delta n^{(i)}(s).$$

$$\delta\Phi[c] = \int_c [V_x(x,y,z)dydz + V_y(x,y,z)dzdx + V_z(x,y,z)dxdy].$$

4. Functionals of surfaces; exceptional points and curves.

5. Variational equations (equations in functional derivatives of form $\Phi'[c|x, y] = F[c|\Phi, x, y]$); the equation for the Green's function; Hadamard's equation.

6. The condition for integrability; self-adjointness in the second variation.

7. Partial equations and the theory of characteristics.

8. Relation of variational equations to integro-differential equations.

LECTURE II. COMPLEX FUNCTIONALS IN SPACES OF THREE AND FOUR DIMENSIONS.

1. Complex additive functionals of curves; integrability; isogeneity, $\Phi_1 + i\Phi_2$ being isogenous to $F_1 + iF_2$; conjugateness in the ellipse belonging to the vectors V_1 of F_1 and V_2 of F_2; equations which vectors of Φ_1 and Φ_2 must independently satisfy.

2. Case where V_1 and V_2 are unit vectors parallel to x and y axes respectively.

3. The invariant $H[\Phi, \Phi']$; extension of Green's theorem; theorems on determinateness; special case when the triple product $[V_1 V_2 ds] = 0$ represents an integrable equation; reduction by a transformation to case where V_1 and V_2 are unit vectors parallel to x and y axes.

4. Generalization of Cauchy's theorem for the integral of a function of a complex variable; conjugateness; non-additive functionals.

5. Complex functionals of surfaces in space of four dimensions; integrability; isogeneity; integrals of analytic functions of two complex variables.

Lecture III. Implicit Functional Equations.

1. Volterra's theorem; calculation of the variation; reduction of the implicit functional equation to a linear integral equation.

2. Application to existence theorems.

3. Linear functional as limit of an integral; as Stieltjes integral; as Lebesgue integral.

4. Special forms of the linear integral equation with a parameter; the equations of the third and first kinds.

Lecture IV. Integro-Differential Equations of Bôcher Type.

1. Hypothetical experiments as basis of physics.

2. Bôcher's treatment of Laplace's equation; generalization to curvilinear coordinates; in three dimensions; Poisson's equation; Cauchy and Weierstrass regions.

3. The equations

$$\int_c \lambda \frac{\partial u}{\partial n} ds = 0, \qquad \int_c \lambda \frac{\partial u}{\partial n} ds = \int\int f(x, y) d\sigma;$$

principal solution.

4. Extension of Green's theorem.

5. Calculation of the Green's function for elliptic and parabolic equations; the Green's function and Green's theorem for integro-differential equations of Volterra type, where the variables are not separable.

6. Generalization of hyperbolic differential equation; stair solutions; non-homogeneous equation.

Lecture V. Direct Generalizations of the Theory of Integral Equations.

1. Moore's general analysis; bases and closure properties; generalization of Fredholm theory, of Hilbert-Schmidt theory; mixed linear equations.

2. Commutativity; associative combination; permutability.

3. The complex operation; convergence properties; differential properties.

4. Volterra's theory of permutable functions; fundamental rôle of the Volterra relation; fractions of composition; logarithms of composition; symbolical treatment of integro-differential equations.

II.

Lecture I. One-Dimensional Analysis Situs.

1. Analysis situs has two chief divisions, dealing (1) with continuity considerations and (2) with combinatorial ones. These lectures are to deal primarily with (2), making enough use of (1) to give substance to (2).

2. The 0-cell, 0-dimensional manifold (or point-pair), 1-cell, 1-dimensional manifold (or curve or circle) and the 1-dimensional complex, C_1, or linear graph.

3. Separation and sense in a 1-cell and a circle. Continuous transformation and deformation. Theorems on point sets, iteration, etc.

4. The matrix $\| \eta_{ij} \|$ of a C_1. The equations (modulo 2) determined by this matrix and the interpretation of their solutions as circuits in C_1.

5. If C_1 is connected there exists a linearly independent set of $\alpha_1 - \alpha_0 + 1$ simple circuits on which all others are linearly dependent. If $\alpha_1 - \alpha_0 + 1 = 0$, the graph is a tree.

6. Introduction of sense-relations, the matrix $\| \epsilon_{ij} \|$, and the corresponding equations. The number of solutions in a complete linearly independent set is again $\alpha_1 - \alpha_0 + 1$.

7. Notice of special investigations of linear graphs.

Lecture II. The n-Dimensional Complex.

8. The n-cell, n-dimensional sphere, n-dimensional complex C_n. Closed and open or bounded complexes, n-dimensional circuits, n-dimensional manifolds M_n.

9. The matrices $\| \eta^1_{jk} \| = \mathrm{H}_1, \cdots, \| \eta^n_{jk} \| = \mathrm{H}_n$ of a C_n.

The sets of equations (modulo 2) determined by $H_1, \cdots, H_n$ and the interpretation of their solutions as circuits. $H_i \cdot H_{i+1} = 0$ $(i = 1, \cdots, n-1)$.

10. Definition of the numbers $R_0, R_1, \cdots, R_n$.

$$\alpha_0 - \alpha_1 + \cdots + (-1)^n \alpha_n$$
$$= 1 + (-1)^n + (R_0 - 1) + \cdots + (-1)^n \cdot (R_n - 1).$$

If C_n determines a manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (R_i - 1).$$

11. Sense relations and the Poincaré matrices $\| \epsilon_{j,k}^1 \| = E_1, \cdots, \| \epsilon_{j,k}^n \| = E_n$. Discrimination between one and two-sided complexes. The sets of linear equations corresponding to $E_1, \cdots, E_n$. Congruences and homologies. $E_i \cdot E_{i+1} = 0$ $(i = 1, 2, \cdots, n-1)$.

12. The Betti numbers $P_0, P_1, \cdots, P_n$ as defined by Poincaré. For any complex,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=0}^{n} (-1)^i (P_i - 1).$$

For a two-sided manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (P_i - 1).$$

For a one-sided manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + \sum_{i=1}^{n-1} (-1)^i (P_i - 1).$$

13. Theorems on matrices whose elements are integers. Reduction of $E_1, \cdots, E_n$ to normal form. Poincaré's coefficients of torsion. Relation between $R_0, \cdots, R_n, P_0, \cdots, P_n$, and the coefficients of torsion.

Lecture III. Manifolds of n Dimensions.

14. Singular complexes in a manifold M_n. Every closed C_k in an n-cell bounds a C_{k+1}.

15. The numbers R_i and P_i and the coefficients of torsion are the same for all complexes C_n into which an M_n may be subdivided, and hence are the same for all manifolds with

which M_n is in $(1-1)$ continuous reciprocal correspondence. Alexander's proof.

16. Dual complexes. Duality relation satisfied by the constants R_i for all manifolds and by P_i and the coefficients of torsion for two-sided manifolds. The characteristic, $\sum_{i=0}^{n}(-1)^i\alpha_i$, is zero if n is odd.

17. One-to-one correspondence between manifolds. Homeomorphism, homoömorphism, and internal transformation. Application of the theorems of separation of an n-cell.

18. Combinatorial proof of the theorems of separation of an n-cell by polyhedra. The theorem of Jordan and its generalizations.

Lecture IV. Two-Dimensional Manifolds.

19. Reductions of the complex defining a two-dimensional manifold. Normal forms and classification of surfaces.

20. The group of a two-dimensional manifold. Representation in the parabolic and hyperbolic planes.

21. Transformations and equivalence of curves on a surface. Homotopy and isotopy.

22. Continuous transformations and deformations. Theorems of Tietze.

Lecture V. The Group of a Complex.

23. The group defined by any complex or manifold.

24. Generating operations and essential relations. Theorem of Tietze.

25. The Cayley color diagram as developed by Dehn.

26. The group of a knot according to Dehn.

27. Relation to the older theories of knots.

Lecture VI. Three-Dimensional Manifolds.

28. Reductions of the complex defining a three-dimensional manifold.

29. A three-dimensional manifold as one cell with a singular boundary.

30. The Heegard diagram.

31. Any manifold may be decomposed into four cells without singularities.

32. The problem of classifying three-dimensional manifolds.

33. Particular three-dimensional manifolds. Riemann spaces. Poincaré spaces.

VIRGIL SNYDER.

NOTE ON THE ORDER OF CONTINUITY OF FUNCTIONS OF LINES.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, September 4, 1916.)

IT has been proved that if a linear function of a line has continuity of the zeroth order, it can be expressed as the limit of a sequence of definite integrals,* as an integral of Stieltjes,† or as a Lebesgue integral.‡ As has been remarked by Bliss,§ many of the functions occurring in the calculus of variations do not have such continuity. The object of the present note is to show that if a function $U[y(x)]$ is linear and has continuity of the nth order, and if $y(x)$ is of class $C^{(n)}$, then $U[y(x)]$ is equal to the sum of a linear function of $d^n y(x)/dx^n$ which has continuity of the zeroth order, and a function of the values of $y(x)$ and its derivatives at an end point of the curve considered.

The proof is very simple. If $y(x)$ is of class $C^{(n)}$ it can be expressed as

$$y(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!}(x-a)^{n-i}.$$

Then, since $U[y(x)]$, is linear,

$$U\left[\begin{matrix} b \\ y(x) \\ a \end{matrix}\right] = V\left[\begin{matrix} b \\ y^{(n)}(x) \\ a \end{matrix}\right] + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!} U\left[\begin{matrix} b \\ (x-a)^{n-i} \\ a \end{matrix}\right];$$

where the function

$$V[y^{(n)}(x)] = U\left[\int_a^x dx_1 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n\right]$$

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* Hadamard, "Leçons sur le Calcul des Variations," p. 299.

† Riesz, *Annales Scientifiques de L'Ecole Normale Supérieure*, vol. 28 (1911), p. 43.

‡ Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 140.

§ Bliss, *Proc. Nat. Acad. Sciences*, vol. 1 (1915), p. 173.

is seen to be linear and continuous of the zeroth order, and $U[(x-a)^{n-1}]$ is a constant.

To illustrate this, the function $U[y(x)]$ will be taken as the first variation of the integral

$$J = \int_a^b f(x, y, y')dx,$$

when $y(x)$ is replaced by $y(x) + \epsilon\eta(x)$. That is,

$$U[\eta] = \epsilon \int_a^b (f_y\eta + f_{y'}\eta')dx.$$

If $f(x, y, y')$ and $y(x)$ are of class $C^{(2)}$, and $\eta(a) = \eta(b) = 0$, this may be written

$$U[\eta] = \epsilon \int_a^b \left(f_y - \frac{d}{dx}f_{y'}\right)\eta dx.$$

If $f(x, y, y')$ and $y(x)$ are only of class C', this equation is not valid. In this case

$$U[\eta] = V[\eta'] = \epsilon \int_a^b \left(f_{y'} - \int_a^x f_y dx\right)\eta' dx,$$

which is linear and continuous of the zeroth order. It is evident that the expression

$$f_y - \frac{d}{dx}f_{y'}$$

is the Volterra derivative of the integral J considered as a function of $y(x)$, and it is approached with the second order.* Similarly

$$f_{y'} - \int_a^x f_y dx$$

is the derivative of J considered as a function of $y'(x)$. It is approached with the zeroth order.

This can easily be extended to functions of surfaces. For convenience it will be supposed that the function $U[z(x, y)]$ is linear and continuous of the second order, and that $z(x, y)$ is of class $C^{(2)}$ and defined over the region $(0 \leqq x \leqq 1;$ $0 \leqq y \leqq 1)$. It follows immediately that

* For a definition of the order of approach see Fischer, *Amer. Jour. of Mathematics*, vol. 35, no. 4 (1913), p. 383.

$$z(x, y) = \int_0^x dx_1 \int_0^y z_{xy}(x_1, y_1)dy_1 + \int_0^y dy_1 \int_0^{y_1} z_{yy}(0, y_2)dy_2$$
$$+ \int_0^x dx_1 \int_0^{x_1} z_{xx}(x_2, 0)\, dx_2 + z_y(0, 0)y + z_x(0, 0)x + z(0, 0).$$

Proceeding as before,

$$U[z(x, y)] = V[z_{xy}(x, y)] + V_1[z_{yy}(0, y)] + V_2[z_{xx}(x, 0)]$$
$$+ z_y(0, 0)U[y] + z_x(0, 0)U[x] + z(0, 0)U[1],$$

where V, V_1 and V_2 are linear and continuous of the zeroth order and $U[y]$, $U[x]$ and $U[1]$ are constants.

COLUMBIA UNIVERSITY,
June, 1916.

THE EQUATION OF A PLANE RATIONAL CURVE DEFINED BY PARAMETRIC EQUATIONS.

BY HERBERT WILLIAM RICHMOND.

THE explicit equation $F(x, y, z) = 0$ of a rational plane curve, defined by the parametric equations

$$x : y : z = A(t) : B(t) : C(t)$$

where $A(t)$, $B(t)$, $C(t)$ are polynomials of order n in t, is expressed by Salmon (Higher Plane Curves, § 44) as a determinant of $3n$ rows. In the April number of the BULLETIN Professor J. E. Rowe exhibits this equation as a symmetrical determinant of n rows, in which each element is a linear function of x, y, z. It has been my custom when lecturing upon algebraic geometry to obtain this form of equation as follows:

Let (x, y, z) be the coordinates of the point whose parameter is t, and let s be any value of the parameter; then

$$\begin{vmatrix} x & A(t) & A(s) \\ z & C(t) & C(s) \\ y & B(t) & B(s) \end{vmatrix} \quad (= \Delta \text{ say})$$

vanishes for every value of s. Imagine the determinant expanded and the factor $s - t$ removed. We have now a

polynomial in s of order $n - 1$, which vanishes for all values of s; the n coefficients of powers of s severally vanish. But each coefficient is a polynomial in t of order $n - 1$, every power of t being multiplied by a linear function of x, y, z. These powers of t may be eliminated by a determinant of n rows, which is the equation of the curve.

The linear function of (x, y, z) which in the rth row and sth column is the coefficient of $s^{n-r}t^{n-s}$ in the expanded form of $\Delta/(s - t)$, and is readily written down if need be. Just as Salmon's method is an adaptation of Euler's (or Sylvester's) method of eliminating a variable from two polynomials, so the foregoing is an adaptation of Cayley's statement of Bezout's method.

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SCIENTIFIC METHOD IN PHILOSOPHY.

Our Knowledge of the External World as a Field for Scientific Method in Philosophy. By BERTRAND RUSSELL, M.A., F.R.S., Late Fellow of Trinity College, Cambridge. Chicago, London, The Open Court Publishing Company, 1914. vii + 245 pp.

THE philosophic method in science is a thing familiar enough, in sound if not in fact. But what, pray, can be meant by "scientific method in philosophy"? That is what Mr. Russell essays to make clear, in outline at all events, in this book, and he does it partly by a general critique on the methodologies of philosophy and science and partly, indeed mainly, by handling certain specific problems of philosophy in the spirit and the manner of science. The chief ones among these illustrative examples belong to the general problem of the relation between the data of sense and the time, space, and matter of mathematical physics. The discussion is guided by a highly important and exceedingly difficult aim. The aim is to sketch a method available in all departments of philosophy and "adequate to yield whatever objective scientific knowledge it is possible to obtain." Traditional philosophy, however worthy of studious attention, has been too impatient; it has been too ambitious; its pretensions and claims have been grossly extravagant; it has not

been sufficiently disinterested; it has been too subjective, too personal, colored and often vitiated by individual tastes and temperaments. Such defects, which keep the great philosophic systems of the past from being scientific, are to be cured or avoided by means of a right philosophic, or scientific, method. What is this method? It is the method of logical analysis—a method first completely exemplified by Frege, a method copiously illustrated in the first three volumes of Whitehead and Russell's Principia Mathematica and, so far as the great conception of the world of physics as a construction rather than an inference is concerned, to be elaborately and precisely presented by Whitehead in the fourth volume of that work.

Mr. Russell's book, which is composed of the eight Lowell Lectures delivered by the author in Boston in 1914, is a rough, general, semi-popular sketch of the method in question. The presentation does not pretend to be accurate in all its details but it is confident of being sound in general and it is animated by the conviction that the method outlined "has great rewards to offer—triumphs as noteworthy as those of Newton and Darwin—and as important in the long run for the moulding of our mental habits."

The initial lecture, which is entitled "Current tendencies," is devoted to a clearing of the ground. The current tendencies dealt with are three: the classical tradition, evolutionism, and what Mr. Russell, for want of a better name, calls logical atomism. These are in conflict. The third is born of the modern critical movement in mathematics and is the type of philosophy that the author advocates. The first and the second types contend with one another and they are both of them antagonists of the third. The classical tradition seeks "to adapt to present needs the methods and results of the great constructive philosophers from Plato downwards," thus owning such names as Kant and Hegel and represented in our own day by the Appearance and Reality of Mr. Bradley. It is doomed by its too great reliance on pure reason and the meagreness of its imagination in conceiving the universe. The universe as revealed by modern science it does not know. It is dying at the hands of discovered natural facts. Mr. Russell would not, I suppose, deny that this philosophy has performed an immense service in testing the resources of reason. The experiment had to be made, for there was no

way of knowing a priori that reason, which had succeeded in building an immortal geometry, was not competent to construct a true philosophy of the entire world.

Evolutionism, the regnant philosophy of our time, is a reaction against the classical tradition. It springs from and allies itself with natural knowledge, as in the works of Herbert Spencer and Bergson. But it "is not a truly scientific philosophy, either in its methods or in the problems which it considers." It is not sufficiently disinterested; it is too easy-going; it does not know the severity of logical discipline. To understand the world it is necessary to understand change and continuity. But biology, physics, chemistry, and the like can not enlighten us here. The problem is foreign to natural science. Again, evolutionism is primarily interested in human destiny, but scientific philosophy is not. It is more detached. It is not Baconian. It has no essential or primary concern with questions that the other sciences may answer. It is not tethered to mundane interests. Incidentally it may help, as it may be helped by, the other sciences, but it appeals to none except such as desire, beyond all else, "to escape from intellectual bewilderment." Towards mysticism its attitude is this: It does not deny the genuineness of the mystic's alleged insights and intuitions of truth beyond the world of sense, but when the mystic denies truth to the deliverances of sense and normal perception it submits a challenge, it asks for evidence. And unlike Bergsonianism, it regards the logical analysis practiced by a Cantor or a Frege as superior to the instinct of a wasp or of a duck that will lead a brood of chicklets to a pond.

The second lecture bears the somewhat challenging title, "Logic as the essence of philosophy." In what sense is the implied thesis to be understood or to be justified? Certainly not at all if logic is to be identified with Aristotelian syllogistic. Mr. Russell does not pause to give due credit to the great logical work of Aristotle, possibly because he thinks the world has already done more than justice in that matter. But a great outburst of genius often blinds centuries of men and certainly Aristotle fettered the mediaeval mind. In a measure the fetters were broken by Bacon and Galileo. Yet Bacon's conception of the nature and proper function of induction was superficial and mistaken. Induction can not replace deduction, but can widen its scope. Deduction is not

all of it syllogistic. But how deduce by induction? What principle justifies inference, for example, from past to future sunrises? If, with Mill, we say the law of causation, then we must ask what justifies belief in that law. That the law is known a priori is rendered improbable by the formulation of the law, which runs somewhat as follows: "A causal law is any general proposition in virtue of which it is possible to infer the existence of one thing or event from the existence of another or of a number of others." If we suppose causal law to be an assumption, then evidently it can not justify inference in any fundamental or ultimate way. Is it an empirical generalization? Then, how justify such generalization? Not by enumeration, for this can at best yield probability—whatever that is—, nor by any other empirical means, for generalization transcends experience. Inference from the observed to the unobserved, if valid, must rest, not on any law of causality, but upon the principle of induction. What is this principle? It is hard to state it quite satisfactorily but it is much like this: "If, in a great number of instances, a thing of a certain kind is associated in a certain way with a thing of a certain other kind, it is probable that a thing of the one kind is always similarly associated with a thing of the other kind; and as the number of instances increases, the probability approaches indefinitely near to certainty." The proposition may be false; it admits of neither proof nor disproof. Yet it is the only justification that the kind of inference in question admits of.

But it is not to be admitted, because general propositions can not be empirically established, that we can not know any general propositions to be true. We have such knowledge and it belongs to logic. Logic consists of two parts, and we are now to see wherein and why it is the essence of philosophy. "The first part investigates what propositions are and what forms they may have." It "enumerates the different kinds of atomic propositions," that is, propositions which, when asserted, assert that a certain thing has a certain quality or that certain things have a certain relation; it enumerates the different kinds of "molecular" propositions, or propositions containing "conjunctions—*if, or, and, unless*, etc."; and it enumerates the various kinds of general propositions, such as, all equilateral triangles are equiangular; and so on. For logic the content of propositions is of no importance, but form is

all-important. The second part of logic "consists of certain supremely general propositions which assert the truth of all propositions of certain forms." An example of such supreme propositions is: "If anything has a certain property, and whatever has this property has a certain other property, then the thing in question has the other property." Such a proposition is true, and is known to be true, but can not be established empirically. This second part of logic is the first part of mathematics; it is the first part of logic that makes logic the essence of scientific philosophy. One is reminded of the saying of Leibnitz: "Ma métaphysique est toute mathématique."

It is in the third lecture, "On our knowledge of the external world," that Mr. Russell comes to close grips with his task and best indicates the method of "logical analysis" or the philosophy of "logical atomism" by operating it on certain fundamental problems. From the earliest times philosophy has from various motives found it necessary to discredit sense-data and to suppose something more real behind the veil. And modern science and especially physics (taken in a comprehensive sense) have not escaped the like necessity. Atoms, molecules, electrons, points, instants, time, space (the time and space of mathematical physics)—none of these things and their kind is known or knowable to sense or perception. Neither can they be logically inferred from the data of sense. They can not be proved to exist nor can their non-existence be shown logically. Why assume them? It is not necessary to do so, and, if not necessary, such assumption is forbidden by the principle of Occam's Razor. What, then, is proposed instead? Instead of illegitimately pretending to *infer* such things as the points, instants, matter, time, and space of mathematical physics and instead of *assuming* the existence of such things, it is proposed and shown to be possible to *construct* them from the data furnished by sense. That done, we shall know them to exist in whatever sense we may suppose constructs to exist, because we have made them, and physics will cease to be an illegitimate inference or a structure based on pure assumption and will be instead a structure built of and upon sense data, the only facts immediately known or knowable respecting the external world. How is the task of construction to be performed? A hint, by means of one example, must here suffice.

Let us confine ourselves to the sense of sight and see how we may construct a corresponding space. The sketch, for the want of room, must needs be rough. Each of many minds, looking out from its own view-point, sees at each moment a three-dimensional world. No two minds behold precisely the same thing. Each has its own "perspective," its own "private world." A mind at an intermediate point of view will have an intermediate private world. Such perspectives or worlds may be correlated by the principle of similarity, for whilst no two are identical, they may resemble each other in various degrees. Two that are much alike may be said to be "near" one another in space but this space is entirely different from "the spaces inside the two perspectives." It is a relation, a relation between the two private worlds or spaces. Between these we can in thought intercallate others more and more similar and hence nearer and nearer together. In this way public space, which consists of relations between private spaces, can be made continuous. Thus space is a constructed affair: it is a system, a system of perceived or perceivable private worlds. In ways quite analogous, we can construct a "thing," a "point," "here," "instant," "time" and so on and on. The delineation of the process with critical remarks as to its philosophic significance occupies the fourth lecture, "The world of physics and the world of sense."

The fifth lecture is devoted to "The theory of continuity." The author ranges himself on the side of those who regard sensation as a continuous function of stimulus though he does not think the doctrine capable of being established empirically. As this doctrine involves the necessity of supposing that two sensations may be different despite the fact that their subject can not discriminate them, it appears to involve a contradiction in terms or a verbal quibble or a verbal confusion. It seems sound to say that two sensations that are sensed as the same are the same, *as* sensations.

Then follow two lectures devoted to the notion of infinity—"the problem of infinity considered historically" and "the positive theory of infinity." Though these lectures are intensely interesting, they contain but little, except some novelty of exposition, that is likely to be new to the student acquainted with the literature of Cantorism. This is not to imply that they have not very great value as exhibiting the nature and power of logical atomism as a method and type of philosophy.

The work closes with a lecture treating "The notion of cause, with applications to the free-will problem." The discussion is very acute, the finest in the book. Hume's classical analysis is resumed and shown to require extension and this is undertaken. The vulgar notion of a cause as compelling its effect must be abandoned as having no warrant in logic and no essential rôle in natural science. A cause may as well accompany or follow its effect as precede it. Far more important than the notion of cause is that of causal law, which has been above stated in one form. It is acutely contended that, as causes do not compel, the acts of will may be caused without being externally coerced, and that omiscience, including knowledge of the entire future, is consistent with every thing in freedom that is worth preserving.

The book as a whole must be judged as an important contribution to the science of philosophy even if the reader must remain convinced that much that is destined to continue to be called philosophy will not, through logical analysis or other means, yield solid, scientific, objective knowledge. Personal idiosyncrasies are themselves facts and they are often more interesting than, and quite as important as, generic results that ignore them. In the future, as in the past, the value of philosophy will consist, not wholly in propositions established by it, but largely in philosophizing. Let the two kinds flourish side by side, but let them not be confounded.

CASSIUS J. KEYSER.

SHORTER NOTICES.

Combinatory Analysis. By Major PERCY A. MACMAHON. Volume 1. Cambridge University Press, 1915. 300 + xix pp.

THE author states that "the object of this work is, in the main, to present to mathematicians an account of theorems in combinatory analysis which are of a perfectly general character, and to show the connection between them by as far as possible bringing them together as parts of a general doctrine. Little attempt has been hitherto made either to make a general attack upon the territory to be won or to coordinate and arrange the ground that has been already gained. The combinatory analysis as considered in this work occupies the ground between algebra, properly so

called, and the higher arithmetic. The methods employed are distinctly algebraical and not arithmetical."

Section I is devoted to the study of symmetric functions. The theory is developed with particular reference to the theory of the partitions of numbers. In general we may have in view the symmetric function

$$\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_s^{p_s},$$

the whole function consisting of a number of terms similar to the one attached to the sign of summation; in each term we may take $p_1, p_2, \cdots, p_s$ to be in descending order of magnitude, but the quantities α, therein appearing, will be any s selected from $\alpha_1, \alpha_2, \cdots, \alpha_n$ in any permutation. It thus appears that the succession of numbers $p_1, p_2, \cdots, p_s$ in descending order of magnitude is a sufficient specification of the symmetric function and, if

$$p_1 + p_2 + \cdots + p_s = w,$$

we may regard the function as denoted by the partition $(p_1p_2\cdots p_s)$ of the number w. It is now shown that the general theory of combinations is essentially involved in the algebra of monomial symmetric functions. Every multiplication of monomial symmetric functions involves a theorem in combinations.

Further developments of the theory of symmetric functions are made by introducing the two differential operators of Hammond:

$$d_\lambda = \frac{d}{da_\lambda} + a_1\frac{d}{da_{\lambda+1}} + a_2\frac{d}{da_{\lambda+2}} + \cdots$$

$$D_s = \frac{1}{s!}(d_1^s),$$

where (d_1^s) indicates a symbolic multiplication as usual in Taylor's expansion. In Chapters IV and V it is shown that the algebra of these operators is parallel with the algebra of symmetric functions.

Section II is a generalization of the theory of Section I. The theory of symmetric functions is associated not with a number but with the partition of a number. The theory of Section I is that particular part of a more general theory which is associated with that partition of a number which is

composed wholly of units. Hammond's operators d and D are given an extended field of operation and, in the enlarged theory, are shown to be important instruments for multiplying and evaluating the symmetric functions that present themselves. In Chapter IV binomial coefficients are treated as symmetric functions denoted by partitions with zero parts, and the formation of symmetrical tables involving them is explained.

In Section III the enumeration of combinations and permutations is treated from the point of view supplied by the theory of symmetric functions. In Chapter II a theorem is established termed "a Master Theorem from the masterly and rapid fashion in which it deals with various questions otherwise troublesome to solve. Many illustrations of its power are given. In particular in Chapter III it is shown to supply instantly the solution of the generalized 'problème des rencontres.' In finding expressions for the sum of powers of binomial coefficients it is singularly effective."

Chapter V is devoted to lattice permutations. Consider an assemblage of letters $\alpha^p\beta^q\gamma^r \cdots$ in which the numbers $p, q, r, \cdots$ are in descending order of magnitude. This particular permutation of the assemblage can be denoted by a regular graph consisting of rows of dots. The successive rows will have $p, q, r, \cdots$ dots respectively and the graph is the same as serves to denote the partition $(pqr \cdots)$ of the number $p + q + r + \cdots$. Such a graph may be

$$\begin{array}{l} \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot \end{array}$$

for $p = 6$, $q = 4$, $r = 1$.

If we take *any* permutation of $\alpha^p\beta^q\gamma^r \cdots$ we shall arrive finally at the same graph by proceeding from left to right of the permutation and placing a dot in the first row, or in the second, or in the third according as we reach a letter α, β, or γ, etc.

The author states that lattice permutations "are in themselves interesting but are of importance principally because they are shown in Volume II to supply the key to solutions of certain questions which involve arrangements, under given conditions, of numbers in spaces of two and three dimensions; numbers not arranged at points in a straight line but at the

crossing points of lines which compose lattices in two and three dimensions."

Section IV is devoted to the theory of the compositions of numbers. The word connotes a partition in which account is taken of the order of occurrence of the parts. Thus the partition a, b, c of the number $a+b+c$ would have six compositions abc, acb, bac, bca, cab, cba involving the parts a, b, c. Chapter I treats with unipartite numbers; Chapter II with multipartite numbers, a graphical representation being given; Chapter III with the graphical representation of the compositions of tripartite and multipartite numbers. Chapter IV gives a complete solution of Simon Newcomb's problem: A pack of cards of any specification is taken—say there are p cards marked 1, q cards 2, r cards 3, and so on—and being shuffled is dealt out on a table; so long as the cards that appear have numbers that are in ascending order of magnitude, equality of numbers counting as ascending order, they are placed together in one pack, but directly the ascending order is broken a fresh pack is commenced and so on until all the cards have been dealt. The probability that there will result exactly m packs or at most m packs is required. Chapter V gives a generalization of the foregoing theory.

In Section V the subject of the perfect partitions of numbers is dealt with as a necessary preliminary to the discussion of arrangements upon a chess board. The latter, as involving conditions to be satisfied by the numbers appearing in a row or in a column, are of the magic square nature. Operators and functions are designed with the object of discovering the laws of formation of the systems of diagrams to which they naturally lead. As the simplest example choose the operation d/dx and the function x^n. We observe that if x^n be written out in the form $xxx \cdots x$ the operation is equivalent to writing unity for one of the x's in all possible ways and adding the results together. We obtain nx^{n-1} because one x can be selected in n different ways. So operating n times successively with d/dx we dissect the operation into $n!$ distinct operations and these give rise to $n!$ distinct diagrams. For consider a square of n^2 compartments; we may place a unit at the intersection of the rth row and cth column to indicate that when d/dx was operating for the rth time our process was to substitute unity for the cth x counting from the left. So we obtain $n!$ diagrams which possess the property that n units

appear one in each row and one in each column. We thus by the operation upon the function enumerate the diagrams with this property.

The problem of the Latin square is solved. This problem is to place n different letters $a, b, c, \cdots$ in each row of a square of n^2 compartments in such wise that, one letter being in each compartment, each column involves the whole of the letters. The number of arrangements is required. "The question is famous because, from the time of Euler to that of Cayley inclusive, its solution was regarded as being beyond the powers of mathematical analysis. It is solved without difficulty by the method of differential operators of which we are speaking. In fact it is one of the simplest examples of the method which is shown to be capable of solving questions of a much more recondite character." Most of the operators are those of the infinitesimal calculus and the numbers involved in the partitions of the functions are positive integers, excluding zero. If we admit zero as a part in the partitions, we have to do with the operations of the calculus of finite differences.

In Section VI the theory of distributions is applied to the enumeration of the partitions of multipartite numbers.

The author states that "In the present volume there appears a certain amount of original matter which has not before been published. It involves the author's preliminary researches in combinatory theory which have been carried out during the last thirty years."

The book is not intended for class room work. There is an absence of problems, but plenty of exercise for one who carries out in detail all of the work indicated. There is no index but a full table of contents. There are errors. It is not our intention to catalogue these. We refer to one merely to support our statement: on page 5 last line for $a_2{}^p$ write $a_2{}^{p_2}$. At certain places conditions are implied which we think should be explicitly stated. We mention but one: page 4 line 3 we should have $\lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots + s\lambda_s = s$. Problems concerning symmetry of determinants find no place in the present treatise. There are thirteen pages of tables at the end exhibiting in a condensed form some of the results of the foregoing theory. In general the work is well written and clear.

W. V. LOVITT.

Plane Analytic Geometry, with introductory chapters on the Differential Calculus. By Professor MAXIME BÔCHER. Henry Holt and Company, New York, 1916. vii + 235 pp.

THIS compact and clearly written textbook gives the essential properties of the conic sections and ends with two chapters on the differentiation of algebraic functions with applications.

The lemniscate is the only curve of higher degree which is discussed at any length, but its graph is correct, thereby differing from the representations commonly published. Other curves are barely mentioned, nor has the book an index. Thus the student of engineering especially must wait for a later course to unfold cardioids, catenaries, spirals, and witches.

The preface states that "Analytic geometry, if properly taught, is a difficult subject, and concentration on a few of its important principles is necessary if mastery is the aim," and later, "It is only by degrading it to a course in graphics (curve plotting, numerical problems, etc.), that a course in analytic geometry can be made easy to the average student."

The student will have to be careful to note that in a right-handed system of coordinates, a positive (left-handed) rotation changes the OX axis into the OY axis. In a note the author states that "The word 'normal' (in Hesse's normal form of the straight line equation) is here used in the sense of 'standard.' It has nothing to do, as some American textbooks have implied, with the normal to a curve or line."

Throughout the book, curves with no real points are consistently referred to as "no locus," and definitions are so framed as to exclude degenerate cases. The chapter on determination of loci is short but explicit, while that on the general equation of the second degree gives a detailed and complete discussion, ending with the problem of a conic through five points.

For the student proceeding to the calculus, a course in solid analytic geometry must follow this text, but otherwise one who has finished this book will be excellently prepared.

F. H. SAFFORD.

NOTES.

THE twenty-third annual meeting of the American Mathematical Society will be held at Columbia University on Wednesday and Thursday, December 27–28, 1916, in connection with the meeting of the American association for the advancement of science. At this meeting President ERNEST W. BROWN will deliver his presidential address, the subject of which will be "The relation of mathematics to the natural sciences." Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by December 13. Abstracts to be printed in advance of the meeting should be sent in by December 6.

A NEW edition of the List of Officers and Members of the Society is now in preparation and will be issued early in January. Blanks for furnishing necessary information have been sent to the members. A prompt response will contribute materially to the correctness and completeness of the List.

THE first summer meeting of the Mathematical Association of America was held under the presidency of Professor E. R. HEDRICK at the Massachusetts Institute of Technology, Cambridge, Mass., on Friday and Saturday, September 1–2, thus immediately preceding the summer meeting of the American Mathematical Society. The attendance numbered 126, including 110 members of the Association. The programme included the following papers: E. V. HUNTINGTON: "The teaching of elementary dynamics"; D. E. SMITH: "The history of mathematical recreations"; J. N. VAN DER VRIES: "Combined courses in mathematics for college freshmen"; F. S. WOODS: "Combined courses in mathematics for freshmen in technical schools." An announcement of plans was made by the committee on mathematical requirements, and a preliminary report was received from the bureau of information. The dinner on Friday evening was attended by about 80 members and friends. The Association will hold its next meeting at Columbia University on Friday and Saturday, December 29–30.

THE sixty-ninth meeting of the American association for

the advancement of science will be held in New York during the week December 26–30 under the presidency of Dr. C. R. VAN HISE. Professor L. P. EISENHART is vice-president, and Professor F. R. MOULTON secretary of Section A.

THE British association for the advancement of science held its annual meeting at Newcastle September 5–9, under the presidency of Sir ARTHUR EVANS. Professor A. N. WHITEHEAD was president of Section A: mathematical and physical science. Professor Whitehead's address before the section, on "The organisation of thought," recently appeared in *Science* (September 22).

THE closing (October) number of volume 38 of the *American Journal of Mathematics* contains the following papers: "Conditions for the complete reducibility of groups of linear substitutions," by HENRY TABER; "On sextic surfaces having a nodal curve of order 8," by C. H. SISAM; "A theorem connected with irrational numbers," by W. D. MACMILLAN; "On the representation of arbitrary functions by definite integrals," by W. B. FORD; "Metric properties of nets of plane curves," by H. R. KINGSTON; "On a special elliptic ruled surface of the ninth order," by H. C. GOSSARD.

ON the occasion of his seventieth birthday, December 7, 1915, Professor A. VOSS, of the University of Munich, received from the Munich technical high school the honorary degree of doctor of technical sciences.

PROFESSOR M. PASCH, of the University of Giessen, celebrated the fiftieth anniversary of his doctorate on August 21, 1915. On this occasion the University of Breslau renewed his diploma.

THE birthplace of WEIERSTRASS in Osterfelde in Westphalia has recently been marked by a memorial tablet.

DEAN T. F. HOLGATE, of Northwestern University, has been elected ad interim president of the University.

PROFESSOR J. J. LUCK, of Vanderbilt University, has been appointed adjunct professor of mathematics in the University of Virginia.

MR. C. H. CLEVENGER has been appointed instructor in mechanics and mathematics in the school of mines of the University of Minnesota.

MR. C. N. REYNOLDS has been appointed instructor in mathematics at Wesleyan University.

DR. P. R. RIDER has been appointed instructor in mathematics in Washington University.

MR. J. J. TANZOLA, of Columbia University, has been appointed instructor in mathematics in the U. S. Naval Academy.

AT Dartmouth College, Dr. C. H. FORSYTH, of the University of Michigan, has been appointed instructor in mathematics. Dr. F. J. MCMACKIN has resigned his instructorship in mathematics to engage in high school work.

PROFESSOR F. PRYM, of the University of Würzburg, died December 15, 1915, at the age of seventy-four years.

PROFESSOR P. DUHEM, of the University of Bordeaux, died in September at the age of sixty-seven years.

THE death is announced of Dr. William ESSON, Savilian professor of mathematics in the University of Oxford, at the age of eighty-eight years.

PROFESSOR S. B. MACLAREN, of University College, Reading, died August 14, 1916, from wounds received in battle.

PROFESSOR L. L. CONANT, of the Worcester Polytechnic Institute, was killed in a street accident on October 11, 1916, at the age of fifty-nine years. Professor Conant had been a member of the American Mathematical Society since 1892.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

HALPHEN (G. H.). Œuvres, publiées par les soins de C. Jordan, H. Poincaré, E. Picard, avec la collaboration de E. Vessiot. Tome 1. Paris, Gauthier-Villars, 1916. 8vo. 44 + 570 pp. Fr. 20.00

JONES (W. H. S.). Scientific method in schools: a suggestion. Cambridge, University Press, 1916. 36 pp. 1s.

JORDAN (C.). See HALPHEN (G. H.).

LINEHAN (P. H.). Contributions to equilong geometry. (Diss., Columbia.) Lancaster, Pa., P. H. Linehan, 1916. 4to. 6 + 38 pp. Paper. $1.00

MACAULAY (F. S.). The algebraic theory of modular functions. Cambridge, University Press, 1916. 14 + 112 pp. 4s. 6d.

PICARD (E.). See HALPHEN (G. H.).

POINCARÉ (H.). See HALPHEN (G. H.).

SCHWALLER (R.). Etude sur les nombres. Paris, Librairie de l'Art indépendant, 1916. 12mo. Fr. 1.50

VESSIOT (E.). See HALPHEN (G. H.).

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). Elements of geometry. Parts 5–6. London, Macmillan, 1916. 2s.

BEY (M.). Principes nouveaux de calculs rapides sans l'emploi des logarithmes. Paris, Desforges, 1915. 8vo.

CHILD (J. M.). See BARNARD (S.).

HERVÉ (E. L.). Conversion des mesures étrangères en mesures françaises. Paris, Challamel, 1916. 8vo. Fr. 3.00

LEIGH (C. W.). See PALMER (C. I.).

LEMOINE (A.). Arithmétique. Cours supérieur et complémentaire, brevet élémentaire. Paris, Hachette, 1916. 18vo. Cartonné. Fr. 1.80

LENNES (N. J.). See SLAUGHT (H. E.).

MARSH (H. W.). Interpolated six-place tables of the logarithms of numbers and the natural and logarithmic trigonometric functions. New York, Wiley, 1916. 8vo. 14 + 155 pp. Cloth. $1.25

MILLIS (J. F.). See STONE (J. C.).

PAILHADE (J. DE R.). Sur les avantages et les progrès de la mesure décimale française des angles. Nice, 1916.

PALMER (C. I.) and LEIGH (C. W.). Plane and spherical trigonometry with 5-place tables. 2d edition. New York, McGraw-Hill, 1916. 8vo. 186 + 132 pp. $1.50. Text alone, $1.00. Tables alone, $0.75.

PATERSON (W. E.) and TAYLOR (E. O.). Elementary geometry, theoretical and practical, covering Stage III of the recommendations of the Board of Education Circular 711. Vol. 1, 1914: Triangles and quadri-

laterals; Vol. 2, 1915: Circle and similar figures. Now issued in one volume. Oxford, Clarendon Press, 1916. Cr. 8vo. 328 pp. 3s.

PHILIP (M.). Algebra through simple equations. New York, Hinds, Hayden and Eldredge, 1916. 6 + 52 pp. $0.48

ROGER-COURT (—.). Arithmétique. Cours moyen. Certificat d'études. Livre du maître. Paris, Colin, 1916. 12mo. 590 pp. Cartonné. Fr. 4.00

ROORDA VAN EYSINGA (H.). Cours de mathématiques élémentaires. III: Manuel de géométrie. Théorie et problèmes. Paris, Payot, 1916. 12mo. 194 pp. Fr. 3.00

ROSENBERG (F.). Preliminary geometry. Oxford, Clarendon Press, 1916. 6 + 220 pp. 2s.

SLAUGHT (H. E.) and LENNES (N. J.). Intermediate algebra. Boston, Allyn and Bacon, 1916. 8vo. 249 pp. $0.75

STONE (J. C.) and MILLIS (J. F.). Plane geometry. Chicago, B. H. Sanborn, 1916. 8vo. 10 + 278 pp. $0.85

TAYLOR (E. O.). See PATERSON (W. E.).

III. APPLIED MATHEMATICS.

BAUSCH AND LOMB OPTICAL COMPANY. Metro manual; a handbook for engineers, containing technical information regarding the construction, adjustment and use of transits, tachymeters, theodolites, alidades, levels, etc. (9th and revised edition of Saegmuller vest-pocket handbook.) Rochester, N. Y., Bausch and Lomb, 1915. 199 + 37 pp.

BROWN (J. C.). See FINNEY (H. A.).

BURROUGHS ADDING MACHINE COMPANY. Instructions for operating and using Burroughs adding and listing machines, class three hundred. 9th edition. Detroit, Mich., Burroughs, 1915. 4to. 31 pp.

CARLIER (J.). La résolution de l'équation de traction d'un véhicule en mouvement par le "puissance-mètre." Paris, Le Génie civil, 1915. 8vo.

CONNAISSANCE des temps ou des mouvements célestes à l'usage des astronomes et des navigateurs, pour l'an 1918, publiée par le Bureau des Longitudes. Paris, Gauthier-Villars, 1916. Cartonné. Fr. 4.75

DICK (F. J.). Ancient astronomy in Egypt and its significance. (Papers of the School of Antiquity, University Extension Series, No. 7.) Point Loma, Cal., Aryan Theosophical Press, 1916. 4to. 18 pp.

DUHEM (P.). Le système du monde. Histoire des doctrines cosmologiques de Platon à Copernic. Tome 4. Paris, Hermann, 1916. Royal 8vo. 4 + 597 pp. Fr. 19.00. Edition sur papier de Hollande. Fr. 25.00

FINNEY (H. A.) and BROWN (J. C.). Modern business arithmetic. New York, Holt, 1916. 8vo. 298 pp. $0 85

GRIALOU (J.). Cours d'hydraulique. Paris, Gauthier-Villars, 1916. 8vo. 6 + 455 pp. Fr. 20.00

KARAPETOFF (V.). Engineering applications of higher mathematics. New York, Wiley. 8vo. Part 1. Problems on machine design, 1912. 14 + 69 pp. Part 2: Problems on hydraulics, 1916. 16 + 103 pp. Part 3: Problems on thermodynamics, 1916. 16 + 113 pp. Part 4:

Problems on mechanics of materials, 1916. 4 + 81 pp. Part 5: Problems on electrical engineering, 1916. 7 + 65 pp.
Each part $0.75

MADSEN (V. H. O.). Le service géodésique en Danemark (1816–1916). Copenhagen, Bianco-Luno, 1916. 4to.

MONCKTON (J. H.). Practical geometry. 4th edition. New York, W. T. Comstock, 1915. 8vo. 13 + 82 pp. $1.00

PALMER (A. R.) and STEPHENSON (J.). Commercial arithmetic and accounts. London, Bell, 1916. Part 1: 14 + 292 + 56 pp. Part 2: 11 + 293–514 + 57–154 pp.

RELATORIO ASTRONOMICO. Uma das maiores conquistas de todos os tempos. Lisboa, A. de Mandonça, 1916. 8vo.

SAEGMULLER. See BAUSCH.

SMITH (W. G.). Practical descriptive geometry. 2d edition, enlarged and reset. New York, McGraw-Hill, 1916. 8vo. 10 + 256 pp.
$2.00

STEPHENSON (J.). See PALMER (A. R.).

TRIMBLE (C. J. A.). See USHERWOOD (T. S.).

USHERWOOD (T. S.) and TRIMBLE (C. J. A.). Practical mathematics for technical students. Part 2. London, Macmillan, 1916. Cr. 8vo.
7s. 6d.

Book ID
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AN EXTENSION OF THE THEORY OF NUMBERS BY MEANS OF CORRESPONDENCES BETWEEN FIELDS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 4, 1916.)

1. To each number a of a field or domain of rationality R let correspond a unique number $F(a)$ of R. Define two operations $\oplus$ and $\odot$ on the numbers $F(a)$ by the equations

$$F(a) \oplus F(b) = F(a + b), \quad F(a) \odot F(b) = F(ab),$$

holding for any two equal or distinct numbers a, b of R. These operations obey the commutative, associative, and distributive laws of ordinary addition and multiplication. For example,

$$\{F(a) \oplus F(b)\} \odot F(c) = F\{(a + b)c\}$$
$$= \{F(a) \odot F(c)\} \oplus \{F(b) \odot F(c)\}.$$

The set of numbers $F(a)$ combined by these two operations therefore form a field $F(R)$, whose zero of addition is $F(0)$ and unity of multiplication is $F(1)$.

2. In particular, let R be the domain of all rational numbers and let the coefficients of $F(a)$ be rational. If a, b and $a/b = q$ are all integers, then $F(a) = F(b) \odot F(q)$ and $F(a)$ will be said to be divisible by $F(b)$. Since $a = bq + r$ implies

$$F(a) = \{F(b) \odot F(q)\} \oplus F(r),$$

Euclid's process for finding the G.C.D. of two integers a, b leads to the G.C.D. of $F(a)$, $F(b)$. We call $F(a)$ a prime if its only divisors are $F(\pm a)$ and $F(\pm 1)$. When the preceding equation holds, we say that $F(a)$ and $F(r)$ are congruent modulo $F(b)$. It is now a simple matter to enunciate the analogues, for the numbers $F(a)$ and the operations $\oplus$ and $\odot$, of the theorems in the theory of numbers. If p is a prime not dividing a, then $F(a) \odot F(a) \cdots \odot F(a)$, to $p - 1$ factors, is congruent to $F(1)$ modulo $F(p)$. Again, $F(1) \odot F(2) \cdots \odot F(p - 1)$ is congruent to $F(-1)$. These analogues to Fermat's and Wilson's theorems follow at once

from the latter by the principle of correspondence and need not be proved independently of them. A like remark is true of the reciprocity law and other theorems of the theory of numbers.

3. The advantage of choosing a linear fractional function

$$F(a) = \frac{la + m}{na + t} \quad (lt - mn \neq 0) \tag{1}$$

as our $F(a)$ lies in the fact that the correspondence between the fields R and $F(R)$ is now (1, 1), so that the numbers of R form a field also when combined by the new operations $\oplus$ and $\odot$. We have

$$\alpha \oplus \beta = \frac{(mn^2 - 2ntl)\alpha\beta + l^2t(\alpha + \beta) - ml^2}{-n^2t\alpha\beta + mn^2(\alpha + \beta) + l^2t - 2mnl}, \tag{2}$$

$$\alpha \odot \beta = \frac{(mn^2 + lt^2)\alpha\beta - ml(n + t)(\alpha + \beta) + ml(m + l)}{nt(n + t)\alpha\beta - nt(m + l)(\alpha + \beta) + m^2n + l^2t}. \tag{3}$$

In particular, if $n = 0$, $t = 1$, then $F(a) = la + m$, and

$$\begin{aligned} \alpha \oplus \beta &= \alpha + \beta - m, \\ \alpha \odot \beta &= \frac{1}{l}(\alpha - m)(\beta - m) + m. \end{aligned} \tag{4}$$

The special case of the latter in which l and m are certain expressions in a single parameter was treated by L. Schrutka,* who resorted to computations to prove the associative and distributive laws in this special case and devoted many pages to the proofs of the analogues of theorems in the theory of numbers, without making clear that the results follow at once by correspondence.

4. The operation defined by

$$\alpha \odot \beta = \frac{a\alpha\beta + b(\alpha + \beta) + c}{d\alpha\beta + e(\alpha + \beta) + f} \tag{5}$$

obeys the associative law if and only if

$$be = cd, \quad b^2 + ce = ac + bf, \quad e^2 + bd = ae + df. \tag{6}$$

First, let $a, \cdots, f$ be integers and let $b = b_1d_1$, $c = b_1c_1$, where d_1 and c_1 are relatively prime integers. Then $e = e_1c_1$,

* "Theorie der Polygonalreste," *Monatshefte für Mathematik und Physik*, vol. 16 (1905), pp. 167–192.

$d = e_1 d_1$, where e_1 is an integer. The remaining conditions (6) now hold if and only if $a - e_1 c_1 = q d_1$, $f - b_1 d_1 = - q c_1$, where q is an integer. Next, when $a, \cdots, f$ are any numbers (not necessarily integers) of the field R, the conditions (6) are equivalent to

$$b = b_1 d_1, \quad c = b_1 c_1, \quad e = e_1 c_1, \quad d = e_1 d_1,$$
$$a = e_1 c_1 + q d_1, \quad f = b_1 d_1 - q c_1,$$

where b_1, c_1, d_1, e_1, q are numbers of R. We now have the most general operation (5) under which the numbers of R form a group.

5. We are led to a fraction of the form (5) in which α and β enter linearly if we demand that the inverse operation shall be applicable to every pair of numbers of the field. Suppose that also $\alpha \oplus \beta$ is a similar symmetric function of α and β. If these two operations obey the associative and distributive laws, it seems probable that they must be of type (2) and (3), defined by the linear fractional correspondence (1). This is easily proved for integral functions:

$$\alpha \oplus \beta = A\alpha\beta + B(\alpha + \beta) + C,$$
$$\alpha \odot \beta = a\alpha\beta + b(\alpha + \beta) + c.$$

Of the conditions for $(\alpha \ominus \beta) \odot \gamma = (\alpha \odot \gamma) \oplus (\beta \odot \gamma)$, those which involve γ^2 show that $Aa = Ab = 0$, whence $A = 0$, and the remaining conditions are

$$aC + b = 2Bb, \qquad bC + c = 2Bc + C.$$

By the associative law for $\oplus$, $B^2 = B$, whence $B = 1$. Thus $b = aC$, $c = aC^2 - C$, and we get (4) with $m = - C$, $l = 1/a$.

NOTE ON THE DISTRIBUTION OF QUADRATIC RESIDUES.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society, October 30, 1915.)

THE present note relates mainly to the distribution of quadratic residues for a rational prime modulus. A special quadratic form is also considered.

1. Let the prime be $p = 4n + 1$ and assume that k is the number of distinct positive quadratic residues which are less than $\sqrt{p}$. If x is an integer such that

$$x^{\frac{1}{2}(p-1)} \equiv -1 \pmod{p}, \tag{1}$$

then by a known theorem* it is possible to write

$$x \equiv \pm \frac{m}{n} \pmod{p},$$

where $0 < m < \sqrt{p}$ and $0 < n < \sqrt{p}$. There are $\frac{1}{2}(p-1)$ distinct positive non-quadratic residues of p which are less than p, and from (1) we have

$$\left(\frac{m}{p}\right)\left(\frac{n}{p}\right) = -1.$$

We may then assume

$$\left(\frac{m}{p}\right) = 1, \quad \left(\frac{n}{p}\right) = -1.$$

Let there be l distinct positive quadratic non-residues less than $\sqrt{p}$. If m/n is a quadratic non-residue of p, then $-m/n$ and $\pm n/m$ are also non-residues, and we therefore have

$$kl \geqq \tfrac{1}{8}(p-1). \tag{2}$$

We also have $k + l = [\sqrt{p}]$, where $[\sqrt{p}]$ denotes the largest integer in $\sqrt{p}$. We get, after using (2),

$$l \text{ or } k \geqq \frac{[\sqrt{p}] - \sqrt{[\sqrt{p}]^2 - \frac{1}{2}(p-1)}}{2},$$

which gives

THEOREM I. *If p is a prime of the form $4n + 1$, then in the set $1, 2, \cdots, [\sqrt{p}]$ there are at least*

$$\left[\frac{[\sqrt{p}] - \sqrt{[\sqrt{p}]^2 - \frac{1}{2}(p-1)}}{2}\right]$$

distinct quadratic residues and at least the same number of non-residues.

2. In connection with the distribution of quadratic residues the writer has proved the following:

* This BULLETIN, 1915, p. 61.

THEOREM II. *Let p be a prime of the form $4n+3$ and let pR be the sum of the quadratic residues of p. Consider the $\mu = \frac{1}{2}(p-1)$ integers $h < p$ defined by*

$$h + [ah] < p, \tag{3}$$

where $[ah]$ is the least positive residue of ah modulo p and a is a fixed integer less than $p-1$. Then the number of quadratic residues in such a set is equal to

$$\left\{\left(\frac{a+1}{p}\right) - \left(\frac{a}{p}\right) - 1\right\} R + S + T,$$

where $S = 0$ or μ according as $(a/p) = \mp 1$, and $T = 0$ or μ according as $(a+1)/p = \pm 1$.

The proof of this result will be given in another paper. The theorem evidently shows that if the number of incongruent positive quadratic residues less than $\frac{1}{2}p$ is known then it is possible to write down immediately the number of quadratic residues in any set (3).

3. By a theorem already cited it is possible to express all the $p-1$ incongruent residues of p, which are prime to p, by means of the set

$$\pm \frac{m}{n}, \tag{4}$$

modulo p, where m and n each range over all the distinct positive quadratic residues of p which are $< \sqrt{p}$. We proceed to find another expression for the number of incongruent fractions (4). Reduce each fraction to its lowest terms and let the number of distinct fractions be K, say. If any two are congruent modulo p we must have

$$\frac{m}{n} \equiv -\frac{m'}{n'} \pmod{p},$$

and, since each m and n is $< \sqrt{p}$, we obtain

$$mn' + m'n = p. \tag{5}$$

Corresponding to such a representation of p, we derive the following relations:

$$\begin{aligned} &\frac{m}{n} \equiv -\frac{m'}{n'}, \quad -\frac{m}{n} \equiv \frac{m'}{n'}, \quad \frac{n}{m} \equiv -\frac{n'}{m'}, \quad -\frac{n}{m} \equiv \frac{n'}{m'}, \\ &\frac{m}{m'} \equiv -\frac{n}{n'}, \quad -\frac{m}{m'} \equiv \frac{n}{n'}, \quad \frac{m'}{m} \equiv -\frac{n'}{n}, \quad -\frac{m'}{m} \equiv \frac{n'}{n}, \end{aligned} \tag{6}$$

modulo p. In counting the number of incongruent fractions in the set (4) we must therefore consider the number of representations (5). We shall regard two representations

$$mn' + m'n = p, \quad m_1n_1' + m_1'n_1 = p$$

as the same if and only if $m = m_1$, $n' = n_1'$, $m' = m_1'$, $n = n_1$. If N is the number of representations of this type, then the relations (6) show that

$$N = K - (p - 1).$$

Now K by definition is equal to twice the number of distinct positive irreducible fractions whose numerators and denominators are each not greater than $\sqrt{p}$. Hence*

$$K = 4(\varphi(2) + \varphi(3) + \cdots + \varphi([\sqrt{p}])) + 2,$$

where $\varphi(k)$ denotes the number of integers $< k$ and prime to it. We therefore have

THEOREM III. *If p is a prime, then the number of representations of p in the form*

$$xy + x'y',$$

where x, y, x', y' are all positive integers $< \sqrt{p}$, is equal to

$$-(1 + p) + 4\sum_{k=1}^{[\sqrt{p}]} \varphi(k).$$

PROOF OF A GENERAL THEOREM ON THE LINEAR DEPENDENCE OF p ANALYTIC FUNCTIONS OF A SINGLE VARIABLE.

BY MR. HAROLD MARSTON MORSE.

(Read before the American Mathematical Society, September 5, 1916.)

A PROOF of the following theorem has to my knowledge not been published to date. The theorem contains as a special case the ordinary theorem concerning the wronskian. Its usefulness in a general treatment of single-valued func-

* Lucas, Théorie des Nombres, p. 393.

tions on a Riemann surface by means of abelian integrals of the second kind was pointed out by Professor Osgood in a lecture course just completed at Harvard.

Let there be given p functions, $f_1(t), f_2(t), \cdots, f_p(t)$, analytic in a region S of the t-plane. Consider the p-square functional determinant, the ith row of which $(i = 1, 2, \cdots, p)$ is

$$f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_1-1)}(t_1), \cdots, f_i(t_1), f_i^{(\lambda_2)}(t_2), f_i^{(\lambda_2-1)}(t_2), \cdots, f_i(t_2),$$
$$\cdots, \cdots, f_i^{(\lambda_\mu)}(t_\mu), f_i^{(\lambda_\mu-1)}(t_\mu), \cdots, f_i(t_\mu).$$

We denote this determinant by $D[t_1, t_2, \cdots, t_\mu]$. We shall have occasion to indicate determinants of the type of D by enclosing the ith row, without the subscript i, in two vertical bars.

THEOREM. *A necessary and sufficient condition for the linear dependence of $f_1, f_2, \cdots, f_p$ is that D vanish identically in all of its arguments.*

Consider a determinant $\overline{D}[\bar{t}, t_1, t_2, \cdots, t_\mu]$, obtained from D by replacing the first column of D by a column $f_1(\bar{t}), f_2(\bar{t}), \cdots, f_p(\bar{t})$, where $\bar{t}$ is a variable independent of $t_1, t_2, \cdots, t_\mu$. We will first show that $\overline{D} \equiv 0$ in all of its arguments, if the same is true for D. If $\lambda_1 = 0$, we have immediately that

$$\overline{D}[\bar{t}, t_1, t_2, \cdots, t_\mu] \equiv D[\bar{t}, t_2, \cdots, t_\mu] \equiv 0.$$

If $\lambda_1 > 0$, we will prove that $\overline{D} \equiv 0$ by showing that at any point $t_1, t_2, \cdots, t_\mu$, and for $\bar{t} = t_1$

$$(1) \quad \frac{\partial^n}{\partial \bar{t}^n}\overline{D} = |f^{(n)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \cdots, \cdots,$$
$$f'(t_\mu), f(t_\mu)| = 0 \qquad (n = 1, 2, 3, \cdots),$$

where the last $p - 1$ columns of the determinant of (1) are the same as the corresponding columns of D. Equation (1) holds for $n = 1$, since for $n = 1$ the determinant of (1) either has two columns identical, or else is the determinant D. We proceed to prove by mathematical induction that (1) holds for all values of n. We therefore assume that (1) holds for $n = m$, that is, that

$$(2) \quad |f^{(m)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \cdots, \cdots, f'(t_\mu), f(t_\mu)| \equiv 0$$

in $t_1, t_2, \cdots, t_\mu$. Upon differentiating (2) with respect to t_1, we have

$$(3)\quad \begin{aligned} &|f^{(m+1)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \cdots, \cdots, f'(t_\mu), f(t_\mu)| \\ &\quad + |f^{(m)}(t_1), f^{(\lambda_1)}(t_1), f^{(\lambda_1-2)}(t_1), \cdots, \cdots, f'(t_\mu), f(t_\mu)| \equiv 0, \end{aligned}$$

where the last $p-1$ columns of the first determinant of (3) are the same as those of (1), and the last $p-2$ columns of the second determinant of (3) are the same as those of (1).

To prove the second determinant of (3) equal to zero, consider a matrix made up of the columns of D together with the first column of the determinant of (2). We assume for the present that some cofactor $\bar{A}$ of the elements of the first column of D does not vanish identically. We observe that in the matrix above there are only two p-rowed determinants having $\pm \bar{A}$ as a first minor, namely (2) and D. Whence* any p-rowed determinant of the given matrix vanishes. The latter determinant of (3) is such a determinant. Whence the first determinant of (3) vanishes identically. The induction is complete, and we have that

$$\bar{D}[\bar{t}, t_1, t_2, \cdots, t_\mu] \equiv 0.$$

If $\bar{D}$ be developed with respect to the elements of its first column, and for a set of values of $t_1, t_2, \cdots, t_\mu$, for which the above mentioned cofactor $\bar{A} \neq 0$, we have

$$A_1 f_1(\bar{t}) + A_2 f_2(\bar{t}) + \cdots + A_p f_p(\bar{t}) \equiv 0$$

in $\bar{t}$, where the A's are constants with respect to $\bar{t}$, and not all zero. The proof of the theorem is complete under the assumption that some cofactor $\bar{A} \neq 0$.

In the contrary case, we take in place of D, and from the upper right-hand corner of D, the largest determinant for which not all of the cofactors of the elements of its first column vanish identically. There will be such a determinant unless all the f's vanish identically, in which latter case the f's are obviously linearly dependent. Further such a determinant will be of the same general form as D. It will be identically zero in all of its arguments; for if not, it could not be the largest determinant for which not all of the cofactors of

* If in a given matrix a certain r-rowed determinant is not zero, and all the $(r+1)$-rowed determinants of which this r-rowed determinant is a first minor are zero, then all the $(r+1)$-rowed determinants of the matrix are zero. Cf. Bôcher, Introduction to Higher Algebra, p. 54.

the elements of its first column vanish identically. The proof applied to D is therefore applicable to this determinant and the theorem is proved in general.

HARVARD UNIVERSITY,
June 1, 1916.

NOTE ON THE LINEAR DEPENDENCE OF ANALYTIC FUNCTIONS.

BY DR. G. A. PFEIFFER.

(Read before the American Mathematical Society, September 5, 1916.)

THE theorem proved in the preceding note, that a necessary and sufficient condition that p analytic functions, $f_1(t)$, $f_2(t)$, $\cdots, f_p(t)$, be linearly dependent is that the determinant whose ith row is

$$f_i^{(\lambda_1)}(t_1),\ f_i^{(\lambda_1-1)}(t_1),\ \cdots,\ f_i(t_1),\ f_i^{(\lambda_2)}(t_2),\ f_i^{(\lambda_2-1)}(t_2),\ \cdots,\ f_i(t_2),$$
$$\cdots,\ \cdots,\ f_i^{(\lambda_\mu)}(t_\mu),\ f_i^{(\lambda_\mu-1)}(t_\mu),\ \cdots,\ f_i(t_\mu)$$
$$(i = 1, 2, \cdots, p;\ p = \mu + \sum_{i=1}^{\mu} \lambda_i)$$

vanish identically in $t_1, t_2, \cdots, t_p$, can be readily proved if we assume the fundamental theorem that the identical vanishing of the wronskian of p analytic functions implies their linear dependence.

By rearranging the columns of the determinant of the theorem we obtain the determinant Δ whose ith row is

$$f_i(t_1),\ f_i'(t_1),\ \cdots,\ f_i^{(\lambda_1)}(t_1),\ f_i(t_2),\ f_i'(t_2),\ \cdots,\ f_i^{(\lambda_2)}(t_2),$$
$$\cdots,\ \cdots,\ f_i(t_\mu),\ f_i'(t_\mu),\ \cdots,\ f_i^{(\lambda_\mu)}(t_\mu).$$

Without losing any generality we shall assume that

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_\mu.$$

Now the derivative of order nq of the q-rowed determinant whose ith row ($i = 1, 2, \cdots, q$) is

$$f_i(t),\ f_i'(t),\ \cdots,\ f_i^{(q-1)}(t)$$

is equal to a positive integer times the q-rowed determinant whose ith row is

$$f_i^{(n)}(t),\ f_i^{(n+1)}(t),\ \cdots,\ f_i^{(n+q-1)}(t)$$

plus a sum of q-rowed determinants each of which has at least one column consisting of the derivatives of $f_1(t)$, $f_2(t)$, $\cdots, f_q(t)$ of an order less than n.

Using this fact we find upon differentiating the determinant Δ $(\lambda_1 + 1)(\lambda_2 + 1)$ times with respect to t_2 and putting $t_2 = t_1$ that the result is equal to a positive integer multiplied by the p-rowed determinant whose ith row is

$$f_i(t_1),\ f_i'(t_1),\ \cdots,\ f_i^{(\lambda_1)}(t_1),\ f_i^{(\lambda_1+1)}(t_1),\ \cdots,\ f_i^{(\lambda_1+\lambda_2+1)}(t_1),$$
$$f_i(t_3),\ \cdots,\ f_i^{(\lambda_3)}(t_3),\ \cdots,\ \cdots,\ f_i(t_\mu),\ \cdots,\ f_i^{(\lambda_\mu)}(t_\mu).$$

The other determinants which result from the differentiation drop out when t_2 is put equal to t_1, since each of them then has at least two columns identical. Repeating this process for the other variables in turn, we finally have the p-rowed determinant whose ith row is

$$f_i(t_1),\ f_i'(t_1),\ \cdots,\ f_i^{(p-1)}(t_1)$$

(or the wronskian of the p given functions) vanishing identically if the determinant of the theorem does. The necessity of the condition of the theorem follows immediately as in the proof concerning the wronskian.

HARVARD UNIVERSITY,
July, 1916.

ON THE LINEAR DEPENDENCE OF FUNCTIONS OF ONE VARIABLE.

BY DR. G. M. GREEN.

(Read before the American Mathematical Society, September 5, 1916.)

As is well known, the identical vanishing of the wronskian of p functions of a single variable is a sufficient condition for their linear dependence if the functions are analytic; if, however, they are not analytic the vanishing of the wronskian is not sufficient. The same situation arises in connection with the theorem proved by Mr. Morse and by Dr. Pfeiffer in the present number of the BULLETIN. The former makes explicit use of the analytic character of the functions involved, whereas the theorem proved by the latter may be neatly stated only for analytic functions, if it is to afford a criterion for linear dependence.

The purpose of the present note is to establish a sufficient condition under which the vanishing of the determinant of Morse and Pfeiffer implies linear dependence, in the case of non-analytic functions—real or complex—of a real variable. The proof is like Frobenius's modification for the wronskian theorem, as given by Bôcher.* The theorem for analytic functions of a real or complex variable follows immediately from the general one.

The following seems to be the most convenient form in which to state the theorem for non-analytic functions, although the hypothesis is stronger than it need be. The superscripts of course denote differentiation with respect to the arguments indicated.

Let $f_1(t), f_2(t), \cdots, f_p(t)$ be p functions of the real variable t defined on the interval

$$I: \qquad a \leqq t \leqq b,$$

and possessing in I all derivatives appearing in the determinant

$$D \equiv \begin{vmatrix} f_1^{(\lambda_1)}(t_1), & \cdots, & f_1(t_1), & \cdots, & \cdots, & f_1^{(\lambda_\mu)}(t_\mu), & \cdots, & f_1(t_\mu) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_p^{(\lambda_1)}(t_1), & \cdots, & f_p(t_1), & \cdots, & \cdots, & f_p^{(\lambda_\mu)}(t_\mu), & \cdots, & f_p(t_\mu) \end{vmatrix},$$

where $t_1, t_2, \cdots, t_\mu$ are independent variables on the interval I. Suppose that

1°. *There exist a set of values of $t_2, t_3, \cdots, t_\mu$ for which the $(p-1)$-rowed determinant formed by deleting the first column and the last row of D vanishes for no value of t_1 in I, and*

2°. *For the said values of $t_2, t_3, \cdots, t_\mu$ the determinant D vanishes for every value of t_1 in I.*

Then the functions $f_1(t), f_2(t), \cdots, f_p(t)$ are linearly dependent in I, and in fact

$$f_p(t) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_{p-1} f_{p-1}(t).$$

We shall suppose throughout the discussion that *the set of values of $t_2, t_3, \cdots, t_\mu$ mentioned in the statement of the theorem*

* "Certain cases in which the vanishing of the wronskian is a sufficient condition for linear dependence," *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 139–149, Theorem II.

Professor Osgood noticed that Bôcher's proof could be extended to the theorem in question; without knowing of this, the present writer sent the proof given below to Professor Osgood, who insisted upon its publication.

have been fixed once for all. It will now be convenient to use a new notation, writing D in the form

$$D \equiv \begin{vmatrix} u_1^{(1)}, & u_1^{(2)}, & \cdots, & u_1^{(p)} \\ u_2^{(1)}, & u_2^{(2)}, & \cdots, & u_2^{(p)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_p^{(1)}, & u_p^{(2)}, & \cdots, & u_p^{(p)} \end{vmatrix},$$

each element standing for the element in the corresponding place in the original determinant. Here the superscripts of course do not denote differentiation.

Let D_i denote the cofactor of $u_1^{(1)}$ in D. Then by part 1° of the hypothesis D_p is zero for no value of t_1 in I. It is immediately evident that

$$(1) \quad u_1^{(i)}D_1 + u_2^{(i)}D_2 + \cdots + u_p^{(i)}D_p = 0 \quad (i = 1, 2, \cdots, p),$$

for every t_1 in I. From these relations may be obtained as follows the $p - 1$ relations, for every t_1 in I:

$$(2) \quad u_1^{(i)}D_1' + u_2^{(i)}D_2' + \cdots + u_p^{(i)}D_p' = 0 \quad (i = 2, 3, \cdots, p),$$

where the accents on the D's denote differentiation with respect to t_1. Those of equations (2) for which $i > \lambda_1 + 1$ follow at once from differentiation of the corresponding equations (1), since, if $i > \lambda_1 + 1$, $u_1^{(i)}$, $u_2^{(i)}$, $\cdots$, $u_p^{(i)}$ are all constants. To obtain the first λ_1 equations of the set (2), differentiate with respect to t_1 each of the first $\lambda_1 + 1$ of equations (1), except the first, and subtract from the result the equation immediately preceding it.

Now add equations (2), after multiplying the first of them by the cofactor of $u_1^{(2)}$ in D_p, the second by the cofactor of $u_1^{(3)}$ in D_p, etc., the last by the cofactor of $u_1^{(p)}$ in D_p. The result is the following equation, true for every value of t_1 in I:

$$(3) \qquad D_1'D_p - D_p'D_1 = 0.$$

Since D_p is by hypothesis zero for no value of t_1 in I, we may divide through equation (3) by D_p^2. This gives the equation

$$\frac{\partial}{\partial t_1}\left(\frac{D_1}{D_p}\right) = 0,$$

for every value of t_1. Recalling that the values of t_2, t_3,

$\cdots, t_\mu$ have been fixed, we may therefore infer that

$$D_1 = -c_1 D_p,$$

where c_1 is a constant. In the same way we find that

$$D_2 = -c_2 D_p, \cdots, \cdots, D_{p-1} = -c_{p-1} D_{p-1}.$$

Since the $(\lambda_1 + 1)$th column of D consists of the elements $f_1(t_1), \cdots, f_p(t_1)$, we have the identity in t_1

$$D_1 f_1(t_1) + D_2 f_2(t_1) + \cdots + D_p f_p(t_1) = 0,$$

which by the relations obtained above may be written

$$D_p(-c_1 f_1(t_1) - c_2 f_2(t_1) - \cdots - c_{p-1} f_{p-1}(t_1) + f_p(t_1)) = 0.$$

Then, since for the fixed values of $t_2, t_3, \cdots, t_\mu$ the factor D_p is different from zero for every value of t_1 in I, the expression in parentheses must vanish identically in t_1. This proves the theorem.

An obvious weakening of the hypothesis may be made if any of the λ's be greater than λ_1. In the statement of the theorem, we supposed that all of the derivatives appearing in D exist throughout I. Of course all that need be required, if derivatives of order higher than λ_1 appear in D, is that such derivatives exist at discrete points in I, constituted by some or all of the fixed values $t_2, t_3, \cdots, t_\mu$ mentioned in part 1° of the hypothesis.

The statement and proof of the general theorem, though given above only for a real independent variable, hold also without any essential modification when the functions are analytic and the independent variable complex. From the theorem for this case may be proved at once the theorem given by Mr. Morse, viz., that if D vanishes identically when considered as a function of the μ independent variables $t_1, t_2, \cdots, t_\mu$, the analytic functions $f_1(t), f_2(t), \cdots, f_p(t)$ are linearly dependent. For, if D_p is not identically zero, there must be a set of values of the independent variables, say $\bar{t}_1, \bar{t}_2, \cdots, \bar{t}_\mu$, for which D_p is different from zero. Then for the fixed values $\bar{t}_2, \bar{t}_3, \cdots, \bar{t}_\mu$, D_p is nowhere zero in a certain neighborhood about the point $\bar{t}_1$, but D is identically zero in this neighborhood. Therefore by the general theorem the functions are linearly dependent in this neighborhood, and, being analytic, are linearly dependent throughout their common region of definition. If, however, D_p is identically zero in all its argu-

ments, we may proceed afresh, since D_p is of the same form as D, but contains only the functions $f_2, f_3, \cdots, f_p$. Reasoning as above, we find that these $p - 1$ functions are linearly dependent, and hence also the p given functions, unless the $(p - 2)$-rowed determinant in the upper right-hand corner of D_p vanishes identically. In the latter case we must continue in the same way, until we finally reach a determinant in the upper right-hand corner of D which is not identically zero. But there must be one such, unless $f_p(t)$ is itself identically zero, in which case the given p functions are linearly dependent. The theorem for analytic functions as stated in Mr. Morse's note therefore follows.

It may be of interest to point out that many—if not all—of the theorems on linear dependence in which wronskians or determinants and matrices constructed like wronskians are involved* have their analogues in corresponding theorems in which appear determinants and matrices resembling the determinant D.

HARVARD UNIVERSITY,
July, 1916.

TRANSLATION SURFACES ASSOCIATED WITH LINE CONGRUENCES.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, October 28, 1916.)

§ 1. *Introduction.*

IN a note published in the BULLETIN in 1914† I established an algorism on a class of surfaces associated with line congruences in 3-space, which result by translation from invariants of plane n-lines. It is the purpose of this paper to apply symbolical methods to the study of some properties of these surfaces.

Two non-homogeneous forms of respective orders m, n, in Plücker's line coordinates p_1, p_2, q_1, q_2, r $[r = (pq)]$, considered together, represent a congruence (m, n). For the sake of symmetry let the variables be changed to the homogeneous system

* Such, for instance, as are given by Bôcher, loc. cit., and by Curtiss, *Math. Annalen*, vol. 65 (1908), pp. 282–298.

† BULLETIN, vol. 20, p. 233.

$$\xi_1 = p_1, \qquad \xi_2 = p_2, \qquad \xi_3 = q_1, \qquad \xi_4 = q_2, \qquad \xi_5 = r,$$
$$\xi_6 = 1; \qquad (\xi_5 = \xi_1\xi_4 - \xi_2\xi_3).$$

Then these forms may be written

$$f_m = (\alpha_1\xi_1 + \cdots + \alpha_6\xi_6)^m = \alpha_\xi{}^m = \alpha_\xi{}'^m = \cdots,$$
$$g_n = (\beta_1\xi_1 + \cdots + \beta_6\xi_6)^n = \beta_\xi{}^n = \beta_\xi{}'^n = \cdots.$$

The generating line of (m, n) is

$$\xi_6x_1 - \xi_1x_3 - \xi_3x_4 = 0, \qquad \xi_6x_2 - \xi_2x_3 - \xi_4x_4 = 0,$$

and this generator is contained in an arbitrary plane $w_x = 0$ provided

$$(1) \qquad \begin{aligned} \xi_1w_1 + \xi_2w_2 + \xi_6w_3 &= 0, \\ \xi_3w_1 + \xi_4w_2 + \xi_6w_4 &= 0, \\ \xi_5w_1 + \xi_4w_3 - \xi_2w_4 &= 0. \end{aligned}$$

§ 2. *The Surfaces Associated with* $(m, 1)$.

A canonical form of $(m, 1)$ is furnished by the pair

$$f_m = 0, \qquad \xi_5 + 1 = 0,$$

and if we replace ξ_5 by -1 in f_m the latter becomes the non-homogeneous

$$(\alpha_1\xi_1 + \cdots + \alpha_4\xi_4 + (\alpha_6 - \alpha_5))^m,$$

or, setting $\alpha_6 - \alpha_5 = \alpha_5$ and introducing, for homogeneity, a new ξ_5 $(= 1)$ in place of ξ_6, f_m becomes the quinary form

$$f_{1m} = (\alpha_1\xi_1 + \cdots + \alpha_5\xi_5)^m = \alpha_\xi{}^m = \alpha_\xi{}'^m.$$

From (1) we obtain, after replacing the old ξ_5 by -1, and solving,

$$(2) \qquad \xi_i = \lambda_i\xi_1 + \mu_i\xi_5 \qquad (i = 1, \cdots, 5),$$

where $\xi_5 = 1$, and

$$(3) \qquad \begin{aligned} &\lambda_1 = \mu_5 = 1, \qquad \mu_1 = \lambda_5 = 0, \qquad \lambda_2 = -w_1/w_2, \\ &\mu_2 = -w_3/w_2, \qquad \lambda_3 = w_4/w_3, \qquad \mu_3 = -w_2/w_3, \\ &\lambda_4 = -w_1w_4/w_2w_3, \qquad \mu_4 = (w_1w_2 - w_3w_4)/w_2w_3. \end{aligned}$$

We now eliminate ξ_2, ξ_3, ξ_4 from f_{1m} by the substitutions (2), giving

$$F(\xi_1, \xi_5) = (\alpha_\lambda \xi_1 + \alpha_\mu \xi_5)^m = a_\xi^m = a_\xi'^m$$
$$(\alpha_\lambda = \alpha_1\lambda_1 + \cdots + \alpha_5\lambda_5).$$

The coefficients of F are rational expressions in the planar coordinates $w_1, \cdots, w_4$; hence any invariant I of this binary form translates into a homogeneous quantic $\varphi(w)$ in the w's representing the surface enveloped by the plane $w_x = 0$ when the latter moves so as to intersect the congruence $(m, 1)$ in an m-line which maintains the property represented, projectively, by $I = 0$. Since every invariant of F is a linear combination of products of factors of type

$$(aa') = a_1a_2' - a_2a_1'; \qquad I = \Sigma k(aa')^p(aa'')^q \cdots,$$

we must compute (aa') in order to obtain $\varphi(w)$ explicitly.

We find

$$(aa') = \alpha_\lambda\alpha_\mu' - \alpha_\mu\alpha_\lambda' = \sum_{i=1}^{4}\sum_{j=1}^{5-i}(\alpha\alpha')_{ii+j}(\lambda\mu)_{ii+j},$$

$$(4) \qquad (\alpha\alpha')_{pq} = \alpha_p\alpha_q' - \alpha_q\alpha_p'.$$

THEOREM. *The class (order in $w_1, \cdots, w_4$) of the translation surface $\varphi(w) = 0$ equals twice the weight of the invariant I.*

In proof, it is evident from (4) that the order of φ in $\lambda_1, \cdots, \lambda_5, \mu_1, \cdots, \mu_5$ is twice the weight of I; but these quantities λ, μ, when reduced to the common denominator w_2w_3, have numerators of dimensions 2 in $w_1, \cdots, w_4$. Nevertheless, from (3), we calculate

$$(5) \quad \begin{aligned} &(\lambda\mu)_{12} = -w_3/w_2, \qquad (\lambda\mu)_{13} = -w_2/w_3, \\ &(\lambda\mu)_{14} = (w_1w_2 - w_3w_4)/w_2w_3, \qquad (\lambda\mu)_{15} = 1, \\ &(\lambda\mu)_{23} = (w_1w_2 + w_3w_4)/w_2w_3, \qquad (\lambda\mu)_{24} = -w_1^2/w_2w_3, \\ &(\lambda\mu)_{25} = -w_1/w_2, \qquad (\lambda\mu)_{34} = -w_4^2/w_2w_3, \\ &(\lambda\mu)_{35} = w_4/w_3, \qquad (\lambda\mu)_{45} = -w_1w_4/w_2w_3. \end{aligned}$$

Hence the theorem is proved.

The explicit symbolical form of $\varphi(w)$ is manifest from (4) and (5) in a form free from adventitious factors;* in fact (aa') is

* If I is the discriminant of F the order of $\varphi(w)$ is $2m(m-1)$. Non-symbolical methods used in my former paper suggested $4m(m-1)$ for this order, but the present theory shows that there is always an adventitious factor (a product of powers of $w_1, \cdots, w_4$) reducing it to $2m(m-1)$. That no further reduction is possible is shown by an example in my previous paper.

proportional to

$$
\begin{aligned}
W = {} & (\alpha\alpha')_{24}w_1^2 + (\alpha\alpha')_{13}w_2^2 + (\alpha\alpha')_{12}w_3^2 + (\alpha\alpha')_{34}w_4^2 \\
(6) \qquad & + (\alpha\alpha')_{14}(w_3w_4 - w_1w_2) - (\alpha\alpha')_{23}(w_3w_4 + w_1w_2) \\
& + (\alpha\alpha')_{25}w_1w_3 - (\alpha\alpha')_{15}w_2w_3 + (\alpha\alpha')_{45}w_1w_4 \\
& - (\alpha\alpha')_{35}w_2w_4,
\end{aligned}
$$

and (aa'') differs from W only by the replacement of $(\alpha\alpha')$ by $(\alpha\alpha'')$.

The degree of $\varphi(w)$ in the coefficients of f_{1m} equals the number of symbols in I.

For illustration, if $m = 2$ the only invariant of $F(\xi_1, \xi_5)$ is its discriminant $(aa')^2$. We may write the quinary form f_{12} as follows:

$$f_{12} = \Sigma a_{ij}\xi_i\xi_j = \alpha_\xi^2 \qquad (i, j = 1, \cdots, 5;\ a_{ij} = a_{ji}).$$

Squaring W we readily compute $\varphi(w)$, here a surface of class 4 and degree 2, viz.,

$$
\begin{aligned}
\varphi(w) = \tfrac{1}{2}W^2 = {} & (a_{22}a_{44} - a_{24}^2)w_1^4 \\
& + 2(a_{14}a_{24} - a_{44}a_{12} + a_{23}a_{24} - a_{22}a_{34})w_1^3w_2 + \cdots.
\end{aligned}
$$

§ 3. *The Congruence* (m, n).

Assuming the linear form of the pair representing the congruence $(m, 1)$ in canonical form $\xi_5 + 1$ enabled us to derive for the result of the elimination (2) a binary form F whose invariants represent projective properties of the m-line in which $w_x = 0$ cuts $(m, 1)$. We now show that the corresponding elimination in the case of a plane cutting (m, n) gives a pair of ternary forms whose simultaneous invariants represent projective properties of the mn-line of intersection.

Solution of equations (1) with $\xi_5 \neq -1$, gives

$$(7) \qquad \xi_i = \lambda_i\xi_2 + \mu_i\xi_4 + \nu_i\xi_6 \qquad (i = 1, \cdots, 6),$$

where

$$
\begin{gathered}
\lambda_1 = -w_2/w_1, \quad \lambda_2 = 1, \quad \lambda_3 = 0, \quad \lambda_4 = 0, \\
\lambda_5 = w_4/w_1, \quad \lambda_6 = 0, \\
\mu_1 = 0, \quad \mu_2 = 0, \quad \mu_3 = -w_2/w_1, \quad \mu_4 = 1, \\
\mu_5 = -w_3/w_1, \quad \mu_6 = 0, \\
\nu_1 = -w_3/w_1, \quad \nu_2 = 0, \quad \nu_3 = -w_4/w_1, \quad \nu_4 = 0, \\
\nu_5 = 0, \quad \nu_6 = 1,
\end{gathered}
$$

and, employing (7) to eliminate ξ_1, ξ_3, ξ_5 from f_m, g_n, there result the respective ternary forms

$$f_{1m} = (\alpha_\lambda \xi_2 + \alpha_\mu \xi_4 + \alpha_\nu \xi_6)^m = a_\xi{}^m = a_\xi{}'^m = \cdots,$$

$$g_{1n} = (\beta_\lambda \xi_2 + \beta_\mu \xi_4 + \beta_\nu \xi_6)^n = b_\xi{}^n = b_\xi{}'^n = \cdots,$$

where, for example,

$$\alpha_\lambda = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_6\lambda_6.$$

Any invariant I of the pair of forms f_{1m}, g_{1n} is a linear combination of products of third order determinants of the two types

$$(aa'a''), \qquad (aa'b).$$

Evaluating the typical case of the second type, we deduce, by a known theorem on determinants of arrays,

$$(8) \qquad (aa'b) = \begin{vmatrix} \alpha_\lambda & \alpha_\mu & \alpha_\nu \\ \alpha_\lambda' & \alpha_\mu' & \alpha_\nu' \\ \beta_\lambda & \beta_\mu & \beta_\nu \end{vmatrix} = \sum_{i,j,k} (\alpha\alpha'\beta)_{ijk}(\lambda\mu\nu)_{ijk}$$

$$(i, j, k = 1, \cdots, 6;\ i < j < k,\ (\alpha\alpha'\beta)_{ijk} = |\alpha_i\alpha_j'\beta_k|).$$

The number of terms in the summation in (8) is ${}_6C_3 = 20$. Substitution from (8) etc. in

$$I = \Sigma k(aa'a'')^p(aa'b)^q \cdots$$

gives a homogeneous form $\varphi(w)$ of degree in the coefficients of f_{1m}, g_{1n}, jointly, equal to the number of symbols occurring in I.

We now prove that *the class of this surface equals twice the number of determinant factors* $(aa'b)$, $\cdots$ *occurring in any term of* I. This number equals the power of the determinant of the collineations in the invariant relation for I, *i. e.*, the index. It is called the weight of I by some writers, and, with this terminology, we now require the proof of the precise theorem stated in § 2, for the present ternary case and congruence (m, n).

Calculating the 20 determinants $(\lambda\mu\nu)_{ijk}$ and writing, for brevity, $(\lambda\mu\nu)_{ijk} = (i, j, k)$, we have

$$(1, 2, 3) = w_2w_3/w_1^2, \qquad (1, 2, 4) = -w_3/w_1,$$

$$(1, 2, 5) = w_3^2/w_1^2, \qquad (1, 2, 6) = 0,$$

$$
(9)\quad
\begin{aligned}
(1, 3, 4) &= -\, w_2 w_4 / w_1^2, & (1, 3, 5) &= 0,\\
(1, 3, 6) &= w_2^2 / w_1^2, & (1, 4, 5) &= w_3 w_4 / w_1^2,\\
(1, 4, 6) &= -\, w_2 / w_1, & (1, 5, 6) &= w_2 w_3 / w_1^2,\\
(2, 3, 4) &= w_4 / w_1, & (2, 3, 5) &= -\, w_3 w_4 / w_1^2,\\
(2, 3, 6) &= -\, w_2 / w_1, & (2, 4, 5) &= 0,\\
(2, 4, 6) &= 1, & (2, 5, 6) &= -\, w_3 / w_1,\\
(3, 4, 5) &= w_4^2 / w_1^2, & (3, 4, 6) &= 0,\\
(3, 5, 6) &= w_2 w_4 / w_1^2, & (4, 5, 6) &= -\, w_4 / w_1.
\end{aligned}
$$

Since the non-vanishing fractions in $w_1, \cdots, w_4$ all have second order numerators and a common denominator w_1^2, the theorem is proved. Substitution of these results in (8) and the results from (8) in I gives the explicit form of the translation surface $\varphi(w)$, in a form free from extraneous factors.

It is obvious that a complete set of invariants gives, in the present case of the congruence (m, n) or in the previous *binary* case of $(m, 1)$, a fundamental system of translation surfaces. For the congruence (2, 2), cut by a plane in a quadrilateral, the complete system consists of four surfaces $(aa'a'')^2$, $(bb'b'')^2$, $(aa'b)^2$, $(abb')^2$, all of degree 3 and class 4.

UNIVERSITY OF PENNSYLVANIA.

PAPPUS. INTRODUCTORY PAPER.

BY DR. J. H. WEAVER.

(Read before the American Mathematical Society, April 24, 1915.)

ONE of the most wholesome tendencies in the study of mathematics today is the desire to give increased attention to the history and genesis of the subject. This tendency has led to a more careful study of the works of the old Greek mathematicians. Of these Pappus of Alexandria was among the last, and from the point of view of the historian one of the most important because it is in his works that we have the only authentic account of the lost works of a large number of preceding mathematicians.

Of the life of Pappus we know practically nothing. Even the date of his activity is known only within the compass of about a century. Suidas places him as a contemporary of Theon of Alexandria (379–95 A.D.), while a tenth century manuscript has a marginal note connecting him with the reign of Diocletian (284–305 A.D.).*

As a writer, Pappus must have been quite versatile if the following list of works attributed to him is any indication:

(1) Description of the World. (2) Comments on the Four Books of the Almagest.† (3) Interpretation of Dreams. (4) On the Rivers of Libya. (5) Commentary on the Analemma of Diodorus.‡ (6) Comments on Euclid's Elements. (7) Comments on Ptolemy's Harmonica. (8) Collection.

Of all these the only one extant even in part is the Collection, which is a summary in eight books of the principal works of preceding Greek mathematicians with comments and lemmas on the works in question. A brief outline of the contents of the Collection is as follows:

Books I and II probably contained an account of the arithmetic of the Greeks. However, all of Book I and part of Book II have been lost. The portion of Book II that remains discusses the methods of multiplication used by Apollonius of Perga.

Book III takes up the geometric side of mathematics. It consists of four parts.§

1. Discussion of the problem of inserting two mean proportionals between two given lines, to which form Hippocrates had reduced the duplication problem.

2. Development of the ten means in use among the Greeks.||

* In this connection see article on Pappus by Sir Thomas L. Heath, Encyclopedia Britannica, 11th ed.

† This title is given by Suidas, but he is in error on this point, because there are thirteen books instead of four in the Almagest of Ptolemy.

‡ Of the Analemma we know nothing. Pappus is the only writer, so far as I know, who mentions it. See Pappus Alexandrinus Collectio, ed. Hultsch, Berlin, 1876–8, p. 246. Hereafter we will refer to this work as "Pappus."

§ Heath, in the article mentioned in the first note above, gives five divisions, the fifth being a second discussion of the duplication problem. I am classifying these two parts as one.

|| The means signify the arithmetic, geometric, harmonic, and seven other means closely allied to them. For a discussion of these see article by J. S. Mackay, "Pappus on the progressions," *Proc. Edinburgh Math. Society*, vol. 6, p. 48.

3. Some theorems on the sums of lines drawn from points on the base of a triangle and included by the sides of the triangle, and the extension of these to polygons.

4. Inscription of the five platonic bodies in a sphere.

Book IV contains:

1. Some miscellaneous theorems, most of which have a bearing on tangent circles.

2. A discussion of the properties of the conchoid, the quadratrix and the spiral of Archimedes and their applications to the three famous problems of Greek geometry.

Book V deals with the theory of isoperimetric figures both plane and solid. It consists of three parts:

1. Theory of plane isoperimetric figures.

2. Theory of solids as developed by Archimedes.

3. Comparison of the five platonic bodies when they have equal surfaces.

Book VI discusses some of the more difficult theorems in the minor works on astronomy.*

Book VII is probably the most important one of the eight, for here we have set forth in a careful systematic fashion the fundamental ideas of the Greeks on loci. Just what the contents of this book cover can be summed up in the following paragraph from the introduction.

"This is the order of the books on loci. One book of the Data of Euclid, two books of Proportional Section, two of Spatial Section, two of Determinate Section and two on Contacts, all by Apollonius, three books of the Porisms of Euclid, two books on Inclinations by Apollonius,† two books on Plane Loci and eight on Conics by the same author, five books of Solid Loci by Aristæus, two books of Surface Loci‡ by Euclid and two books on Means by Eratosthenes. There are then in all thirty-three books of which, as far as the conics of Apollonius, I have set forth the contents for your inspection and I have stated the number of loci, of determinations and

* The Almagest of Ptolemy was known as the major work on astronomy and all others were classed as minor. Of these works Pappus gives lemmas on the Spherics of Theodosius, the Moving Sphere of Autolycus, Days and Nights by Theodosius, Size and Distances of the Sun and Moon by Aristarchus, the Optics of Euclid, and the Phænomena of Euclid.

† For a discussion of the term νεῦσις or inclinations see Works of Archimedes, ed. Heath, Cambridge, 1897, Chap. V.

‡ Whether this work discussed surfaces of the second degree only, or sections of these surfaces also is not definitely known. For a discussion of this point see Heiberg, Studien über Euklid, Leipzig, 1882, pp. 79–83.

cases in each book and have set up quite a few lemmas which are required, nor do I think I have omitted any question in the discussion of these books."*

Book VIII deals with some mechanical problems and a few theorems of pure geometry.†

Although we are not sure what theorems were discovered by Pappus, the following are a few which may be due to his genius:

1. Generalization of the pythagorean theorem.‡
2. Several of the fundamental theorems relating to perspectivity.§
3. A theorem showing the relation between the quadratrix and the spiral of Archimedes.
4. Description of a spiral on a sphere and the quadrature of the area of the surface included between this spiral and a great circle of the sphere.‖
5. Statement of the so-called theorems of Guldin. These are found without proof in the introduction to book VII. Their statement there is as follows: "Figures generated by a complete rotation about an axis have a ratio compounded from the rotating figures and lines similarly drawn to the axes from the centers of gravity of the rotating figures: and figures generated by an incomplete revolution have a ratio compounded from the rotating figures and from the arcs which the centers of gravity of the rotating figures describe."¶
6. Proof of the invariancy of the cross ratio under a projective transformation.

The best edition of the Collection is that of Hultsch** which is in Greek with a parallel Latin version. Sir Thomas L.

* Pappus, p. 636.

† For a further discussion of the contents of the Collection see the article of Heath mentioned in the first footnote on p. 128. Also see Account of the Life and Writings of Robert Simson, by W. Trail, Bath, 1812, pp. 131–180, and Hutton's Philosophical and Mathematical Dictionary, 2d ed., vol. II, London, 1815, pp. 146–48.

‡ For a statement and proof of this theorem see Teaching of Geometry, D. E. Smith, Boston, 1911, p. 263.

§ See in this connection articles by R. C. Archibald, "Centers of similitude of circles and certain theorems attributed to Monge. Were they known to the Greeks?" *Amer. Math. Monthly*, vol. 22, pp. 6–12, and "Historical note on centers of similitude of circles," *Amer. Math. Monthly*, vol. 23, pp. 159–61.

‖ For a discussion of theorems 3 and 4 see Chasles, Aperçu Historique, Bruxelles, 1837, pp. 28 and ff.

¶ Pappus, p. 682.

** See the third footnote on p. 128.

Heath did intend to bring out an English edition, but a recent private letter from Heath speaks for itself: "I do not think that there is much prospect now of my bringing out any book on Pappus, at all events for a long time . . . if I ever have the time to do it at all." The writer has spent the last fifteen months making a careful translation of the Collection, during which time he has noticed that most writers on Pappus have been either extensive or intensive but not both at the same time. In particular he has noticed that most writers on Pappus seem to think that he was guilty of appropriating work that did not belong to him. However, the facts which have come out of this extended study seem to set at naught all such accusations.

First let us turn to a problem that has been most widely discussed, the trisection of an angle. Gow* says that in Book IV Pappus claims as his own a solution of this problem which is doubtless the one that Proclus ascribes to Nicomedes. The solution in question is Prop. 32, which reads as follows: "Let there be any acute angle ABC and let any line AC be drawn perpendicular to BC, and let the rectangle $ACBZ$ be completed, and let ZA be extended to E, which is so assumed that BE being drawn, the segment of it cut off between the lines AC and AE is double the segment AB. Then angle $EBC = 1/3$ angle ABC."

Cantor seems to have some doubts in his mind as to whether this was the solution of Nicomedes effected by means of the conchoid, but does not produce any evidence to support his doubts.†

Hoppe criticizes Cantor for even considering that this solution was the one of Nicomedes and supports his criticisms by pointing out the fact that the insertion of the line DE was accomplished by means of the intersection of a hyperbola and a circle, and that in this connection no mention is made of the conchoid.‡

R. C. Archibald states that the discovery of the application of the conchoid to the trisection problem was claimed by Pappus.§ Sturm says that Pappus seems to claim for him-

* Short History of Greek Geometry, Cambridge, 1884, p. 310.

† M. Cantor, Vorlesungen über Geschichte der Mathematik, 3d ed., Leipzig, 1907, vol. I, p. 354.

‡ Hoppe, Mathematik und Astronomie im klassischen Altertum, Heidelberg, 1911, p. 308.

§ *Amer. Math. Monthly*, vol. 21 (1914), p. 347.

self this application.* The only reference that is cited from the works of Pappus by the above authorities is the following: "We (Pappus) in the Commentary on the Analemma of Diodorus used this curve to trisect an angle."† And from this statement they draw the conclusion that Pappus claims to be the first to apply the conchoid to the trisection problem. But Pappus was not the first to make the application, and does not even claim to be the first, as the following quotation clearly shows. "Nicomedes solved this problem (duplication problem) by means of the conchoid, by means of which he also trisected an angle."‡ This quotation disposes of the question of the claims of Pappus and confirms the statement of Proclus that Nicomedes used the curve to solve both the duplication and trisection problems.§

Also Pappus is accused of proving Prop. 10, Book VIII, in a manner only accidentally different from that of Heron of Alexandria and that he here assumes credit that rightfully belongs to Heron.|| To meet this accusation a translation of the introduction to the proposition in question will be sufficient. It is as follows: "To the same kind of reasoning¶ belongs the problem that a given weight may be moved by a given power. This is found in the Mechanics of Archimedes, in which he is led to say exultingly: 'Give me where I can stand and I will move the world.' Then Heron of Alexandria has clearly explained this in his book called βαρουλκός . . . this being assumed, that the diameter of the wheel shall have to the diameter of the axle the ratio 5 : 1, the weight to be moved shall be 1,000 talents and the power which moves it 5 talents. Now let it be proved by us in the ratio 2 : 1 and let the weight be 160 talents and the power 4 talents."**

In the above two instances the evidence is clear that Pappus was not guilty of appropriating work that belonged to others,

* A. Sturm, Das Delische Problem, Linz, 1895–7, p. 80.

† Pappus, p. 246.

‡ Pappus, p. 56.

§ Proclus, ed. Friedlein, p. 272.

|| See Gow, Short History, p. 310, Cajori, History of Elementary Mathematics, New York, 1914, p. 86, and Ball, Short History of Mathematics, 2d ed., New York, 1893, p. 101. Cajori and Ball do not mention this case specifically, but since it is one of the three current accusations I am assuming that they had this instance in mind.

¶ The reasoning referred to is the reasoning in Prop. 9, in which Pappus calculates in general terms the power that will be required to move a weight in a horizontal direction on an inclined plane.

** Pappus, p. 1060.

but in the third case there is no direct evidence in the works of Pappus himself. However, he is accused of claiming as his own fourteen theorems on isoperimetric figures which Theon of Alexandria says were found in a book written by Zenodorus.* Nokk even goes so far as to claim that all the theorems recorded by Pappus on isoperimetric figures were given by Zenodorus in his book on the subject. Hoppe criticizes Nokk very severely and takes the opposite extreme by stating that Pappus probably knew nothing about Zenodorus and worked out the theorems independently.†

The proofs of the theorems mentioned by Theon are almost identical with the corresponding ones in Pappus. This fact along with some other evidence leads Hultsch to think that Theon borrowed part of his commentary from Pappus.‡

But aside from the above criticisms the following items have a bearing on the situation.

(1) The object of the Collection was to set forth the theories that had been previously developed and not to develop a new theory.

(2) There is a possibility that Pappus may have mentioned the name of Zenodorus, for considerable mutilation has taken place in Book V. Proposition 9 has been lost entirely and Proposition 7 has a portion of its proof incorrectly written as we now have it.

(3) In comparing Apollonius and Euclid Pappus states that it is necessary to be kindly disposed toward any one who can advance mathematical theory even a little.§

(4) In the light of the above discussion this is the only instance in the entire Collection as to which there is any doubt about the appropriation of material, and the omission of a single name (if it was originally omitted) is, in a work the size of the Collection, hardly cause enough for an accusation of theft.

These things seem to the writer to be sufficient to give Pappus the benefit of the doubt, and to clear his name from the stigma that has clung to it for so long. At least an accusation of theft seems hardly justifiable until we have some more evidence on the subject.

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* See in this connection Pappus, pp. 1189–1211, Gow, Short History, pp. 271 and 310, Cajori, History of Elementary Mathematics, p. 86, and Nokk, Abhandlung über Isoperimetrische Figuren, Progr., Freiburg, 1860.

† Mathematik und Astronomie, p. 313.

‡ Pappus, vol. III, p. xv.

§ Pappus, p. 676.

On the other hand the following is a notable instance where Pappus is given credit in some quarters for a thing that he does not even claim as his own. Gow* and Taylor† attribute the discovery of the focus and directrix definition for the conic sections to Pappus. Ball qualifies his statement by saying that the discovery is due to Pappus if it was not stolen by him.‡

Cantor and Chasles mention the property but make no statement other than that it is not found in Apollonius.§ Heath states that Pappus gives us the earliest recorded use and proofs of the properties in question but that the discovery was not due to Pappus,‖ while Zeuthen seems to think that the focus at least for the parabola was known to Euclid.¶

In the Collection there are two references to the properties mentioned above. The first is Prop. 34, Book IV, which reads as follows: "*Some* by other reasoning have explained the trisection of an angle without inclinations.** Let the arc be cut in any ratio for there is no difference whether we cut an arc or an angle. Let it be done and let arc $BC = 1/3$ arc AC and let AB, AC and BC be drawn. Then $\angle ACB = 2\angle BAC$. Let CD bisect $\angle ACB$ and let the lines DE and BZ be drawn perpendicular to AC. Then $AD = DC$ and $AE = EC$. Therefore the point E is given."

Then by means of a very simple proof we have the following relation

$$BZ^2 + ZC^2 = 4EZ^2,$$

from which Pappus draws the following conclusion.

"Now because the points E and C are given and the perpendicular BZ is drawn and given and the ratio $EZ^2 : (BZ^2 + ZC^2)$ is given the point B is on a hyperbola."

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* Short History, pp. 252 and 309.

† C. Taylor, Introduction to the Ancient and Modern Geometry of Conics, Cambridge, 1881, p. xxxvi.

‡ Short History, 2d ed., New York, 1893, p. 101.

§ Cantor, Geschichte, vol. I, p. 452, and Chasles, Aperçu Historique, p. 44.

‖ See the first footnote on p. 128.

¶ Geschichte der Mathematik im Alterthum und Mittelalter, Kopenhagen, 1896, p. 211. See also in this connection Zeuthen, Die Lehre von den Kegelschnitten im Alterthum, Kopenhagen, 1886, pp. 212 and ff., where an extended account of the properties is found, and Encyclopadie der Mathematischen Wissenschaften, Band III, Heft I, pp. 12 and 52–59. This last gives extensive references relative to the development of the focus and directrix.

** See the third footnote on p. 129.

Now according to the introductory statement of the theorem Pappus was only recording the solution of others, but we have here everything concerning the focus and directrix definition except the definite statement of the theorem, and since Pappus by means of the constant ratio was able to say that the point B was on a hyperbola the distinction must have been known.

In the second reference we have the explicit statement and proof of the properties.*

Proposition 238, Book VII, reads as follows: "Let there be a line AB and a point C in the same plane and from a point D let DC be drawn and also DE perpendicular to AB and let the ratio $CD : DE$ be given. Then the point D is on a conic, and if the ratio $= 1$ it is on a parabola, if > 1 on a hyperbola and if < 1 an ellipse."

The proof of this theorem depends upon the proof that the ratio $EZ^2 : (BZ^2 + ZC^2)$ of Prop. 34, Book IV, is given and known. But this is the theorem that Pappus assumes to be known. This consideration along with the fact that Prop. 238, Book VII, is a lemma on Euclid's Surface Loci gives some weight to the statement of Zeuthen mentioned above. This much is evidently true. The idea was not new to Pappus even if we concede that he was the first to put it in definite form.

One thing more ought to be emphasized in any discussion relative to the Collection and that is its remarkable suggestiveness. In order to understand this, one has only to turn to the works of such men as Chasles,† Günther,‡ Descartes,§ Newton,‖ and Steiner,¶ for in the writings of these men it has furnished the basic ideas for analytic geometry, projective geometry, and other allied theories. And if it had so much to offer these men, it ought to furnish some suggestions to the careful reader of today.

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* Pappus, pp. 1005–1015.

† Aperçu Historique, pp. 28 and ff.

‡ "Ueber eine merkwurdige Beziehung zwischen Pappus und Kepler," Biblioteca Mathematica, 1888, p. 81.

§ La Géometrie de René Descartes, Paris, 1886.

‖ See Ball, Short History, 2d ed., p. 355.

¶ Steiner, Werke, herausgegeben von Weierstrass, pp. 81 and ff.

SHORTER NOTICES.

Guida allo Studio della Storia delle Matematiche. By GINO LORIA. Milano, Hoepli, 1916. Pp. xvi + 228. Price 3 lire.

No one can pick up such little volumes as those of the Sammlung Göschen, Sammlung Schubert, Teubner's Aus Natur und Geisteswelt and Mathematische Bibliothek, and the Manuali Hoepli without a feeling somewhat akin to envy. That economic conditions are, or at least have been until now, such as to allow the sale of many of these works for a few cents abroad, while in this country it is practically impossible to publish similar manuals, always seems unfortunate from the standpoint of scholarship, even if excusable from that of labor. So when we see a book like Professor Loria's selling for less than sixty cents, we must all feel regret that it can be accessible only to students who are familiar with the Italian language.

It is now nearly nine years since Professor Loria expressed, at the fourth International Congress of Mathematicians, his belief in the advisability of placing before students a guide for their study of the history of mathematics. His well-known interest in the subject, his contributions to its literature, his courses in the University of Genoa, and his editorial experience, all have tended to fit him for writing such a work, and he is entitled to the thanks of all mathematicians for the care he has shown in its preparation.

The "Guida" is divided into two parts, the first relating to the preparation for research, and the second to aids in research itself. Under the former title are considered the questions of historic method in general, of the bibliography of the history of mathematics, and of the material to be found in periodical literature. In the second part Professor Loria considers the material to be found outside the general histories, not only that which may be characterized under the head of guides to original sources, such as works on the reading of manuscripts, but such bibliographical and biographical works as bear upon the subject in hand. The topics include Greek, Roman, and Oriental biography and bibliography; biography of modern mathematicians in various countries; collections of

works of great mathematicians; catalogue material; reviews such as those in the *Jahrbuch über die Fortschritte der Mathematik;* and the method of using all this material.

Under each of these heads is given a list of books relating to the subject, often with a valuable note concerning an important work, so that the student is assisted in weighing the value of any book that he is likely to consult.

It will at once be seen that here is an aid to the study of the history of mathematics that will be serviceable to all students of the subject and that will save a large amount of dictating and directing on the part of those who give instruction. Whether or not the bibliography is as good as one could wish, it is still a decided help and as such it should be received. It is now nearly seventy-five years since such a work was first attempted in a small way, by De Morgan, but this effort was buried in the British Almanac and Companion for 1843, so that it is rarely seen to-day; and after all, there is little comparison between the two publications.

One of the simplest tasks of a reviewer is to pick flaws in such a work. The mass of material is now so great that it is almost impossible for a single writer to digest it all, especially in the case of monographs and articles hidden in the reports of associations and in periodicals not directly concerned with the history of mathematics. In considering the question of completeness this reviewer first went to his card catalogue and then to his bibliographical notes to see what was on his own shelves that was not listed. It was soon apparent that Professor Loria had simply made a selection, although the basis of this selection was not always apparent. Naturally the Italian writers are more fully considered than those of other countries, but numerous names are missing, presumably because they represent articles which originally appeared in periodicals to which a general reference is made. But whether all these periodicals are named, it is difficult to say, there being only an index of names of authors. There seems to be no reference to the *Comptes rendus* (Paris), for example, although this publication has numerous articles of a historical nature, and so it is possible that Italian references of the same kind are wanting.

The English bibliography is naturally less complete than that of Italy and France. For example, there is no mention of the following works: Anderson, edition of the Arenarius

(London, 1784); Harris, Lexicon Technicum (London, 1716); Wallis, the English edition of his algebra (London, 1685) which, while not as complete as the Latin, is more accessible to most readers; Raphson, History of Fluxions (London, 1715), a valuable work in spite of its bias; Woodhouse, on Isoperimetrical Problems (Cambridge, 1810), a helpful work on the history of the calculus of variations; Hutton, Tracts on Mathematical Subjects (London, 1812), worthy of mention as well as his dictionary; Taylor, translation of Bhaskara's Lilavati (London, 1816); Rigaud, on the first publication of the Principia (Oxford, 1838); Dacier, the English edition of his life of Pythagoras (London, 1707); Stewart and Minto, Life and Writings of John Napier (Perth, 1787); Trail, Life and Writings of Robert Simson (Bath, 1812); Bruce, Memoir of Charles Hutton (Newcastle, 1823), and many similar works. It may, of course, be said that such a name as Hutton is not international; but for English and American students, at least, many titles like the above would be needed. There are also four rather remarkable articles prepared for encyclopedias but often found printed separately, namely, Playfair's General View of the Progress of Mathematical and Physical Science (Edinburgh, 1816); Peacock's article on Arithmetic (London, 1825), justly held as one of the most scholarly treatments of the history of arithmetic in the English language, and his report on the recent progress of analysis (London, 1834); and John Herschel, Mathematics (Edinburgh, 1830), with some good historical material.

In French there are less frequent omissions, and yet a number of important works, some of them doubtless reprints, are wanting. Such, for example, are Biot's Analyse des Ouvrages originaux de Napier relatifs à l'Invention des Logarithmes (Paris, 1835); De Coste, La Vie du R. P. Marin Mersenne (Paris, 1649); Baillet, La Vie de M. Descartes (Paris, 1691, with an English edition in 1693); Dupin, Essai historique sur Monge (Paris, 1819); Savérien, Dictionnaire universel de Mathématiques (Paris, 1789), with considerable historical material; Bertrand's Rapport sur les Progrès de l'Analyse (Paris, 1867); and numerous minor works like those of Boyer (1900) and Bioche (1914).

There are also missing such works as Voss, De quatuor Artibus (Amsterdam, 1650), containing a chronologia mathematicorum not to be despised; Ramus, Scholæ Mathematicæ,

with considerable historical matter; Weidler, De Characteribus Numerorum (Wittembergæ, s.a.); Biering, Historia Problematis Cubi duplicandi (Copenhagen, 1844); Brugsch, Numerorum apud veteres Ægyptios Demoticorum Doctrina (Berlin, 1849); and Budaeus, De Asse et Partibus eius Libri V (Paris, 1514, and various other editions), works of no great value, except the last one, but still important enough to make it worth the while of students to consult them.

Such a list of omissions could, of course, be greatly amplified, and no doubt M. Eneström will see that this is done when the publication of *Bibliotheca Mathematica* is resumed. It would be possible also to mention several unimportant misprints, but this matter, too, may well be left for the careful if sometimes caustic pen of the Stockholm critic. The omissions are not mentioned by way of criticism, because no book of this size can be expected to give more than a limited selection from all the works upon the subject, even from the rather important ones; but they are given simply for the purpose of calling attention to the fact that students must not feel that the list given by Professor Loria is exhaustive. The book is merely a guide which points the general way, and the student must expect to supplement it at every step of his progress. Looked at in such a spirit, the book is a very welcome addition to our literature.

DAVID EUGENE SMITH.

Fundamental Conceptions of Modern Mathematics—Variables and Quantities. By ROBERT P. RICHARDSON and EDWARD H. LANDIS. The Open Court Publishing Company, Chicago and London, 1916.

THE volume under review contains Part I: on Variables and Quantities, and a portion of Part X: on Functional Relations, of the division on Algebraic Mathematics of the treatise entitled Fundamental Conceptions of Modern Mathematics. No mention is made of other divisions of the treatise, but twelve additional parts of the first division are announced. A synopsis of these later parts is given at the end of the present volume and the authors invite "suggestions toward improving the present redaction of these later parts" as well as "comments in criticism of Part I." Some of the topics which are to be treated in the later parts are domains and ranges; limits, bounds, and appanages; symbols, signs, and sigla; differentiation; integration, etc.

The purpose of the work is "to examine critically the fundamental conceptions of mathematics as embodied in the current definitions." The authors complain that "Definitions are laid down only as they are needed for the work in hand, and in their formulation attention is given, not to the needs of mathematical science as a whole, but to those of a single book—too often a book whose sole purpose is to enable more or less stupid youths to pose as graduates of a course in mathematics." One feels that they are inspired by uplift ideals when one reads that "mathematics to-day is indeed far behind most other sciences as regards lucidity of exposition. In a comparatively short time a young man of average ability can become so familiar with chemistry or botany or zoology, as to be able to read intelligently a work in any department of the science whatsoever. But this is not the case with mathematics—a student far above mediocrity, who has taken the best university course in mathematics to be found, will come across mathematical works as unintelligible to him as Chinese or Choctaw. It is not merely that he finds himself unfamiliar with the theorems proven in such works: this would be neither surprising nor detrimental; but he will not even be able to understand what it is that the theorems are about. And to gain the knowledge requisite for this will not be a matter of consulting a lexicon; but one of hard study for several months. This state of affairs is not, we hold, an unavoidable one due to the peculiar difficulties of mathematics. It is due to the lack of systematization; and in particular to the failure of text-books to give any thorough exposition of the fundamental conceptions of mathematics." It behooves the mathematician to take notice of such a thesis and to examine sympathetically any contribution made by its positor. The statements that "the thirst for so-called 'original research,' and the credit attached to it has led mathematicians to disregard such matters" and that "in many quarters the impression prevails that there is nothing more to be done at the foundations of mathematics" raise some doubt as to the competence of the authors to speak authoritatively and put the mathematical reader on his guard. He is likely to maintain this attitude when he reads that "This much-needed revision of mathematics ought undoubtedly to be made from a philosophical standpoint, there being constantly maintained rigid adherence to the requirements of a sane metaphysics in the

best sense of the word and to the canons of a sound logic" and when he is shocked by the bad taste of the statement that "in making a provisional list" of the transformations of equations, "we find that the treatment of Vieta was more truly scientific than that of the pygmies who followed him in this field."

The importance of Part I with reference to the work as a whole is made clear by the statement that "the keynote of our work is the distinction we find it necessary to make between quantities, values and variables on the one hand, and between symbols and the quantities or variables they denote or values they represent on the other." Here the authors complain that "mathematicians confuse values and quantities, and again quantities and variables, though not usually values and variables. And they also confuse symbols (and in general expressions) with the things these denote or represent" and that "the tendency to confusion instead of distinction would indeed seem to be growing, and certain mathematicians would avowedly make mathematics entirely a matter of symbolism."

The first 145 pages are devoted to a critique of the definition of a variable as a quantity or as a symbol of some sort. An explicit statement of what the authors consider to be the "proper definition" of a variable is nowhere given: "Any attempt to give a precise account of the definition of the term 'variable' would require a somewhat lengthy consideration of the philosophical theory of the categories, which cannot be given in this place." An idea of what they have in mind may be gained perhaps from the following quotations: "The quantities contemplated together, when a variable is the object of inquiry, compose a class of quantities which we may call a variable-class. But a wide gulf separates the inquiries instituted with respect to variables and all other inquiries instituted with respect to the members of classes composed of quantities;" "the propositions enunciated concerning a variable do not, in the typical cases, treat of the members of the variable-class taken separately; they treat of the mutual relations between the members of the class. And among these relations, the most common is that state of affairs which exists when the variable possesses a limit—a limit being a quantity which may or may not belong to the variable;" "in any inquiry concerning a variable, one of the most im-

portant, though one of the least regarded facts, is the arrangement in order of the quantities which compose the variable;" "we shall say that the quantities of a variable are arranged in order, when every quantity of the variable has had conferred upon it a relation or order with respect to at least one other quantity of the variable;" "to discover what character quantities must possess to be amenable to those inquiries for which a variable is formed, and therefore to be eligible to join in forming a variable, we will inspect the most typical of all of the relations between the quantities of a variable—the relations which subsist when the variable incessantly approaches a quantity, either as a limit or otherwise. The qualifications hereby educed are conformed to by the quantities of every variable, as an exhaustive examination of the variables found in the mathematical sciences will show. For a variable x to incessantly approach a quantity a, it is requisite: first, that there should be a unifarious arrangement of the quantities of the variable; second, that each quantity in the variable should be *nearer* a than is every previous quantity; third, that no quantity of the variable should be equal to or identical with a. The second condition is that relevant to our inquiry. In stating it we use the word *nearer* in a technical algebraic sense. Of two quantities x_1 and x_2, the latter is said to be *nearer* the quantity a than is the former when the difference between x_2 and a is less numerically than the difference between x_1 and a;" "hence, in order that the variable x shall incessantly approach the quantity a, it must be possible to find the difference between every quantity of x and a—in other words, with each quantity of x, it must be possible to either subtract this quantity from a or to subtract the latter from the former. And for these operations of subtraction to be possible, there must be a certain uniformity of character of a and the quantities of x." We shall not indulge in comment upon these statements. Much space is devoted to a classification of quantities into sorts, kinds and varieties, but unfortunately, we do not find anywhere an explicit statement of what "quantity" means. The authors present their views upon vectors and quaternions (comment on these views will doubtless come from specialists in the field) and proceed to build up a theory of negative, imaginary, complex, and hypercomplex numbers. We are told that "of the services rendered by Hamilton to mathematical science, one of the most important has not been

recognized by mathematicians. With the admission of the relations between vectors of the same sort to membership among the quantities of mathematics, there is furnished ample argument to banish forever, to the limbo of doctrines outworn, the tenet so widely taught by mathematicians, even at the present day, that negative, imaginary, and complex 'numbers' are mere symbols." Indeed they hold that "the ordinary algebra of the present day . . . ought to be largely developed as a complanar vector analysis; and it is on such a basis that we deal, not merely with the imaginary quantities but also with the real negatives, the abstract as well as the applicate. The method of introducing the negative real abstract quantities by the sanction of the Principle of Permanence we are constrained to regard as especially unsatisfactory, though this method is used, in formulating algebra as a system, by the most eminent mathematicians. Our own way of dealing with these and with the imaginary abstract quantities is the natural result of not confining our attention to the formal side of algebraic science, but taking into account its matter as well as its form. Those who would develop algebra as a purely formal science, and as nothing more, are satisfied to stop when they have ascribed the origin of the conception of a negative real abstract quantity to such equations as $x + 1 = 0$, $x + 2 = 0$, etc., and the conception of an imaginary abstract quantity to such equations as $x^2 + 1 = 0$, $x^2 + 2 = 0$, etc. But the conceptions attained when one does not look beyond the equations are nothing more than purely formal conceptions of quantities, and he who would master the matter as well as the form of the science must look deeper. He must attain what might be termed *entitative conceptions*. He must find classes of entities adequate to fulfill the conditions fixed by the formal conceptions."

In the next ten pages the authors review and criticize most unmercifully the definitions of a variable given by various authors, including Baire, Gennochi-Peano, Czuber, Burkhardt, Pringsheim, Pierpont, and Russell. Part I closes with a consideration of certain variables. "The most simple of all variables are the ordinary progressions of arithmetic: arithmetical, geometrical, harmonical, etc." Series are considered in the same connection.

The portion of Part X which occupies the last 30 pages of the volume begins with the statement that "the essential

characteristic of a functional relation between variables we hold to be the like order of corresponding quantities in these variables. For two variables, y and x, to be in functional relation, it is necessary and sufficient that there be two or more quantities of x, x_1, x_2, etc., which respectively correspond to y_1, y_2, etc., quantities of y, and that with every two pairs of corresponding quantities x_m and y_m, x_n and y_n, y_m is subsequent to y_n when x_m is subsequent to x_n and vice versa; y_m is prev ous to y_n when x_m is previous to x_n and vice versa; and finally when x_m is neither previous nor subsequent to x_n (*e. g.*, is abreast of it, as may be the case under a multifarious arrangement) y_m bears a like relation of order to y_n, and is neither previous nor subsequent to the latter and vice versa. In the case of three or more variables, x, y, z, etc., the sufficient and necessary conditions are quite analogous." Comment seems quite superfluous here.—After reviewing some of the historical definitions of the notion of function, the authors take up current views of functions by quoting from Dini, Harkness and Morley, Osgood, and Pringsheim, none of which authors succeed in finding favor with our crusaders. The last six pages of the volume are devoted to an exposition of the "errors of Riemann."

While some of the criticisms made by the writers of this book are doubtless well-founded (confusion in the use of the words *same, equal,* and *identical;* in the use of the word *series;* apparently erroneous ascription to Dirichlet of the definition of function usually credited to him), and while some of the ideas they introduce are interesting and perhaps valuable, (confluence and contrafluence, e. g.), it must be clear from the passages quoted that the book does not contribute in any important way to the solution of the problem which it set, viz., the critical examination of the fundamental conceptions of mathematics.

In the first place much space is devoted to dealing in a superficial way with the nomenclature of mathematics, and introducing without reason which seems sufficient to the reviewer, a considerable number of new terms. It is not at all clear for instance, why "negative protomonic abstract non-zero" is less "crude" and less "unsuited to the present state of mathematics" than "negative number." While uniformity in usage is doubtless desirable, it is really not of much consequence whether the term "complex number" shall

or shall not include the reals or the pure imaginaries or both. To make this issue the basis for a tedious quarter of a page and for the statement that "inability to use language with precision seems to be a failing endemic among mathematicians, and Riemann was not immune" under a page heading "the errors of Riemann" is amusing arrogance. The book does not lack other passages of a similar character. It does not seem necessary nor perhaps even desirable to the reviewer that an elementary text in algebra (as Bauer's Vorlesungen über Algebra or Burnside and Panton's Theory of Equations) should introduce the most general notion of function. The authors seem to find a failure to do so sufficient reason to accuse an author of "serene omninescience of the progress made since the middle of the eighteenth century."

In the second place the authors seem to be oblivious of the fact that mathematics intends to be an abstract science and that it is not primarily concerned with the concrete instances from which the abstractions with which it deals may be made. The larger part of the present volume, stripped of the parts irrelevant to its own purpose, is occupied with finding concrete instances for the abstractions number, variable, and function, upon the basis of vector analysis. The classification of quantities into sorts, kinds, and varieties is without significance for the numbers of mathematics, though perhaps useful if one wishes to provide a concrete basis for each of the types of numbers which occur. To do this may be of interest to the pedagogue, but it is without importance for the mathematician and of questionable value for the scientific philosopher, particularly when for the genesis of the vector analysis itself at least the real number system would be requisite. It is not surprising that the authors can not be satisfied with the definitions of variable and function which they find in mathematical books, when we realize that the writers of these books are not dealing with the things their critics seem to be interested in.

And in the third place it appears to the reviewer that the method pursued by this book is entirely inadequate for its purpose. It is recognized generally that in a systematic discussion of the foundations of any division of mathematics, we must begin by knowing precisely what it is we are going to talk about and what our fundamental assumptions are going to be, i. e., we begin by laying down indefinables and postu-

lates. This the authors of this book have not done and the result is that in many places we are at a loss to determine the meaning of the words used, and that, if we make a correct guess at their meaning, we find ourselves confronted with numerous instances of "vicious circle definitions," i. e., definitions which do not define. We call attention to the discussion of sorts on pages 24 and 25 (what does it mean to add two quantities?); to that of kinds on page 27 (when is one quantity equal to, greater than or less than another?); to the discussion of units on page 39 (what is + 1 when we do not yet know what a unit is?) and to the definition (?) of sort on page 76 et seq., as well as to the discussion of the meaning of zero on page 139.

That the authors do not have any respect for authority will perhaps be regarded as a merit by many, although the manner in which this "originality" finds utterance can not fail to irritate the mathematical reader, particularly when positive evidence of a lack of mathematical maturity comes before him. Such evidence is found in the meaningless discussion on page 188 which aims to "make plain to the veriest tyro in mathematics" that the definition of function as given, for instance, in Osgood's Funktionentheorie is "erroneous"; also in the discussion of Riemann's definition of a function of a complex variable on pages 196 and 197.

The reviewer considers the appearance of this book therefore as a distinctly unfortunate occurrence and he hopes that wiser counsel may prevail if the publication of future parts of the work should be under consideration. Poincaré, Russell, and others have done much to bridge the gulf which has separated philosophy and mathematics. Their work has undoubtedly contributed a great deal on the one hand to the appreciation on the part of mathematicians of the problems in the foundations of their science, and on the other hand to the initiation of a new tendency in philosophy. Further work in this field will therefore be received with interest and hope by mathematicians, provided it give clear evidence of the author's competence in both mathematics and philosophy. Unless this be the case, the workers in the two fields will be driven apart, for each group is inclined unfortunately to hold the other group responsible for the acts of any one whom they have reason to suspect of belonging to that group. It is earnestly to be hoped that the mathematicians who may

happen to read the volume under review may not base upon it a general condemnation of the utterances of philosophers concerning mathematics, but will give themselves an antidote in the form of such books as Russell's Scientific Method in Philosophy or Holt's Concept of Consciousness.

ARNOLD DRESDEN.

Homogeneous Linear Substitutions. By HAROLD HILTON, M.A., D.Sc. Oxford at the Clarendon Press. 1914. Pp. 184.

PROFESSOR Hilton's book is a welcome addition to the text-book literature on the subject of linear substitutions. In the preface the author states that he has "attempted to put together for the benefit of the mathematical student those properties of the homogeneous linear substitution with real or complex coefficients of which frequent use is made in the theory of groups and in the theory of bilinear forms and invariant factors."

The first four chapters, comprising a little more than half of the book, are intended to form an introduction to the whole subject. In the first chapter, which is much the longest in the book, the ordinary method of transforming the general substitution into the normal and canonical forms by means of the poles is shown and the simpler properties of symmetric, orthogonal, unitary, and Hermitian substitutions are given. In the second the author gives a very brief account of invariant factors* and develops the second canonical form which is the direct product of substitutions of the type

$$x_1' = x_2, \quad x_2' = x_3, \cdots, x_{r-1}' = x_r,$$

$$x_r' = e_1x_1 + e_2x_2 + \cdots + e_rx_r.$$

In the third chapter devoted to bilinear forms the Hermitian forms play a prominent part.

To the student who comes to the subject for the first time the fourth chapter on Applications will be one of the most interesting in the book. Illustrations from the theory of equations, from differential equations, from the theory of maxima and minima, from geometry, and from mechanics serve to show the wide range of application of the subject.

* Following Bromwich, Hilton uses the term "invariant factor" instead of "elementary divisor."

Not the least interesting of these applications is the development of the theorems of Thompson and Bertrand on the kinetic energy of systems acted upon by impulses.

The last five chapters are somewhat more advanced in character and are concerned principally with the more detailed properties of symmetric, orthogonal, and Hermitian substitutions. Two brief chapters are devoted to permutable substitutions and to families of bilinear forms.

The book contains a large list of examples which, on the whole, serve to illustrate rather than to extend the subject. For the greater part of the examples indications of the methods of solution are given.

The book contains so much material that has not heretofore been available in one place that it may seem out of place to criticize it for what it does not contain. Nevertheless one can not help feeling that the author has taken his title too literally. For example, it contains no hint of the relation of homogeneous to non-homogeneous substitutions, though in many fields, as in projective geometry and theory of functions, the non-homogeneous form is frequently more convenient.

Again, the reader who is familiar with the group theory will wonder why the word "group" has been excluded from the body of the book even though, as the author says, this phase of the subject has been well treated by Burnside. The introduction of even the elements of the group theory would have given to the book a symmetry and a completeness that it does not now have. Moreover, even Burnside's excellent treatise omits much important material that is to be found in the best continental texts, Weber's Lehrbuch der Algebra, for example. This is particularly true of the fundamental theorems relating to invariants of groups of substitutions of which Hilbert's theorem on the finiteness of form systems is the completest generalization. This gap for English-speaking students has recently been filled in part by the publication of Miller, Blichfeldt, and Dickson's Finite Groups.

The treatment of permutable substitutions, covering as it does a scant seven pages, of which nearly two pages are taken up by examples, could well have been amplified. Of the equations of Weierstrass and Dedekind which play an important part in the theory of algebraic integers as well as in the theory of complex number systems no mention is made.

Likewise the beautiful theorem which asserts that two permutable substitutions must have a common pole is omitted.

The author follows Jordan, Klein, and Burnside in writing the accented, i. e., the new, variables on the left, and writes the factors of a product in such order that multiplication of substitutions is effected through columns-by-rows multiplication of the matrices.

In spite of the omission of some things that the reviewer would have liked to see included, the book is a noteworthy contribution to a subject that is of increasing importance to students of mathematics. It is well written, though the condensed notation makes it rather difficult reading for the beginner in the subject.

ERNEST B. SKINNER.

MATHEMATICS AT AN ITALIAN TECHNICAL SCHOOL.

THE regular courses in the Italian technical schools are five years long, most of the work being prescribed. The schedule is not exactly the same in all of them, yet the general scheme followed can be understood by following the plan of instruction in any one of them. For this purpose we choose that at Milan (Reale Istituto Tecnico Superiore di Milano), which includes a two-year course in mathematics, physics, and chemistry and also in Italian and two foreign languages as preparatory to the regular three-year courses in mechanical, electric, civil, and mining engineering, architecture, industrial, physical, and electric chemistry, and a four-year normal course. Candidates for admission must have a diploma from a recognized Italian secondary school or equivalent credentials. In mathematics they must be familiar with plane and solid geometry, plane trigonometry, algebra including determinants, theory of equations and graphical processes, and the elements of projective geometry.

The following outline applies to all the students except those in architecture, who have briefer courses in all the subjects mentioned.

During both terms of the first year there are three weekly lectures and one hour of exercises in plane and solid analytic

geometry (Professor A. Jorini); four lectures and three hours of exercises in mathematical analysis (Professor U. Cisotti); three hours in physics, six in drawing, four (no laboratory) in chemistry, one in Italian literature, and two each in English and German. The first term's work is very similar to that given under the same heading in our American institutions, the main difference being that much less time is given to exercises. But while the internal structure in analytic geometry is similar, there is a marked difference in the work in mathematics as a whole, for during this same term the student also takes a much more comprehensive course in the calculus. It begins with a review of determinants, systems of linear equations, complex numbers, and the idea of functionality. Derivatives of algebraic functions are introduced as soon as the student has completed the geometry of the straight line, and it is at once followed by a generous treatment of maxima and minima. In the treatment of series general questions of convergence are studied, the theorems of Taylor and Maclaurin derived, and the remainder employed. The next chapter is that on geometric applications, including tangents, inflexions, curvature, evolutes, and singular points.

In analytics, the entire second term is devoted to space; the instruction includes an extensive discussion of quadric surfaces, making free use of polar properties. In analysis, we begin with integration as a formal process, then as summation, followed by the usual elementary geometric applications. Then go back to functions of two or more variables, implicit functions and some geometric applications of partial derivatives. Finally, a short survey of elliptic integrals, integration by series, and a glimpse at differentiation and integration under the sign of integration.

In the second year analysis is taught six hours per week during the first semester (Professor U. Cisotti), projective and descriptive geometry is taught throughout the year, three lectures and six hours of drawing per week (Professor C. Capelli), and rational mechanics both terms, four lectures and one hour of exercises per week (Professor B. Caldonazzo). The hours in physics, chemistry, and languages are about the same as for the first year. The number of actual hours per week required of each student is 30 the first year, 38 the second. In analysis, the topics considered are space curves, multiple

integration, differential equations, approximate solutions, Fourier series, functions of a complex variable, conformal representation, and an introduction to vector analysis.

In projective geometry the work is taken up in the sequence and about the extent treated in Cremona's book* followed by reciprocal radii and general quadratic inversion. In descriptive geometry, the course given is about the same as is found in our better institutions, with the addition of further work on axonometry, central perspective, ruled surfaces of orders three and four, and brilliant points and lines.

The course in mechanics begins with vector analysis followed by statics, including center of gravity, then by kinematics of a point and of a rigid body, precession and relative motion, ellipsoid of inertia. After about two months dynamics is introduced, mass, weight, force defined, impulses, kinetic energy, free and restrained vibrations, resonance, gravitation, central motion, Kepler's law, and the centrifugal pendulum being studied. In the second term the topics considered are d'Alembert's principle, potential energy, examples, including loop the loop, stability of equilibrium, dynamics of a rigid body, compound pendulum, Euler's equations, the gyroscope, and various applications. Finally, dynamics of continuous systems, hydrostatics and hydrodynamics, equations of Laplace and Poisson, lines and tubes of flux, gravitational field, potential of spherical strata, etc.

This completes the regular work in mathematics. During the third year courses differ for students in the various branches, but all take the thermodynamics (three hours), and applied mechanics, nine hours per week.

If we include rational mechanics and projective geometry in pure mathematics, but exclude descriptive geometry, thermodynamics and applied mechanics, we see that the Italian technical school student has thirty-six hours per week for one term of pure mathematics, and has had preliminary work at least equal to that required of our most exacting schools. Moreover, a more advanced course in analysis is offered as an elective by Professor G. Vivanti, to be taken in the fourth or fifth year, and similar choices are possible in several branches of applied mathematics.

VIRGIL SNYDER.

* Cremona was professor of geometry at the Milan technical school when he wrote his well-known treatise on projective geometry.

NOTES.

THE concluding (October) number of volume 17 of the *Transactions of the American Mathematical Society* contains the following papers: "On the invariant system of a pair of connexes," by O. E. GLENN; "Algebraic properties of self-adjoint systems," by DUNHAM JACKSON; "On the second derivatives of an extremal-integral with an application to a problem with variable end points," by ARNOLD DRESDEN; "Deformable transformations of Ribaucour," by L. P. EISENHART; "Jacobi's condition for the problem of Lagrange in the calculus of variations," by D. M. SMITH; "On transcendental numbers," by A. J. KEMPNER; "The linear dependance of functions of several variables, and completely integrable systems of homogeneous linear partial differential equations," by G. M. GREEN; "On the factorization of expressions of various types," by HENRY BLUMBERG; "The formal modular invariant theory of binary quantics," by O. E. GLENN.

THE May, June, September and October numbers of the *Proceedings of the National Academy of Sciences* contain the following mathematical papers: "On the foundations of plane analysis situs," by R. L. MOORE; "A general theory of surfaces," by E. B. WILSON and C. L. E. MOORE; "Theory of an aeroplane encountering gusts," by E. B. WILSON; "Differential equations and implicit functions in infinitely many variables," by W. L. HART; "Note on Lucas' theorem," by M. B. PORTER; "A variable system of sevens on two twisted cubic curves," by H. S. WHITE; "A proof of White's porism," by A. B. COBLE; "On certain asymptotic expressions in the theory of linear differential equations," by W. E. MILNE; "On Newton's method of approximation," by H. B. FINE; "Point sets and cremona groups. Part III," by A. B. COBLE; "Some problems of diophantine approximation: a remarkable trigonometrical series," by G. H. HARDY and J. E. LITTLEWOOD; "Newton's method in general analysis," by A. A. BENNETT.

THE Hamburg mathematical society recently decided not to celebrate its two hundred and twenty-fifth anniversary, on account of the war. The society had in 1915 a membership

of 121, including 21 foreign and 9 honorary members. Of these 49 were serving in the army.

AT Queen's University, Kingston, Ontario, Dr. DANIEL BUCHANAN has been promoted from an associate professorship of mathematics to the professorship of astronomy and mathematics.

AT Washington University, Dr. W. H. ROEVER has been promoted to an associate professorship of mathematics, and Dr. G. O. JAMES to an associate professorship of astronomy and mechanics.

AT Lafayette College Dr. W. M. SMITH has been promoted to a full professorship of mathematics.

AT the University of Arizona Professor H. B. LEONARD, of the University of Oregon, has been appointed professor of mathematics. Assistant professor G. H. CRESSE has been promoted to an associate professorship of mathematics.

AT the University of Oregon assistant professor R. M. WINGER has been promoted to a full professorship of mathematics.

DR. J. K. LAMOND, of Wesleyan University, has been appointed professor of mathematics in Pennsylvania College.

AT the University of Pennsylvania Dr. R. L. MOORE has been promoted to an assistant professorship of mathematics.

AT the University of Illinois Dr. A. R. CRATHORNE has been promoted to an assistant professorship of mathematics.

AT Purdue University Drs. W. V. LOVITT and T. E. MASON have been promoted to assistant professorships of mathematics.

DR. W. E. MILNE, of Bowdoin College, has been promoted to an assistant professorship of mathematics.

MR. H. W. MYERS has been appointed professor of mathematics in Huron College.

MR. F. L. SMITH has been appointed instructor in mathematics at Princeton University.

THE death is announced of Professor EVA S. MAGLOTT, who had held the chair of mathematics in Ohio Northern University for the past thirty-two years.

PROFESSOR CLEVELAND ABBE, of the U. S. Weather Bureau, died October 28 at the age of seventy-eight years. Professor Abbe was a member of the American Mathematical Society from 1892 to 1914.

PROFESSOR A. G. SMITH, head of the department of mathematics and astronomy in the University of Iowa died November 5 at the age of forty-eight years.

BOOK catalogues:—Galloway and Porter, Cambridge, England, catalogue 82, mathematical and physical books, about 680 entries.—G. E. Stechert and Company, 151 West 125th Street, New York, Sets of mathematical periodicals, 42 entries.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

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BAHIER (E.). Recherche méthodique et propriétés des triangles et rectangles en nombres entiers. Paris, Hermann, 1916. 8vo. 7 + 266 pp. Fr. 8.50

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CARSLAW (H. S.). Elements of non-euclidean plane geometry and trigonometry. London, Longmans, 1916. 8vo. 12 + 179 pp. 5s.

CHEMIN (O.). See PETERSEN (J.).

GIRAUD (G.). Sur une classe de groupes discontinus de transformations birationnelles quadratiques et sur les fonctions de trois variables indépendants restant invariables par ces transformations. (Thèse.) Paris, Gauthier-Villars, 1916. 4to. 8 + 169 pp.

GLOBA-MIKHAILENKO (B.). Sur quelques nouvelles figures d'équilibre d'une masse fluide en rotation. (Thèse.) Paris, Gauthier-Villars, 1916. 4to. 3 + 79 pp.

HARDY (G. H.). The integration of functions of a single variable. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 2.) 2d edition. 8 + 67 pp. 3s.

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MIELI (A.). Storia generale del pensiero scientifico dalle origini a tutto il secolo XVIII. Le scuole Ionica, Pythagorica ed Eleata (I Prearistotelici, I). Firenze, Libreria della Voce, 1916. 8vo. 16 + 504 pp. L. 12.00

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VIVANTI (G.). Elementi della teoria delle equazioni integrali lineari. (Manuali Hoepli.) Milano, Hoepli, 1916. 16 + 398 pp. Cloth. L. 4.50

II. ELEMENTARY MATHEMATICS.

BRESLICH (E. R.). Second-year mathematics for secondary schools. 2d edition. Chicago, University of Chicago Press, 1916. 12mo. 20 + 348 pp. Cloth. $1.00

EVANS (G. W.) and MARSH (J. A.). First year mathematics. New York, Merrill, 1916. 235 pp. $0.90

HALE (A. W.). Problems and exercises in algebra and geometry. Boston, Heath, 1916. 8vo. 54 pp. Paper. $0.36

HOYT (F. S.) and PEET (H. E.). Everyday arithmetic; two-book course. Boston, Houghton-Mifflin, 1916. $0.40 + 0.40

MARSH (J. A.). See EVANS (G. W.).

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PEET (H. E.). See HOYT (F. S.).

REMICK (B. L.). See STRATTON (W. T.).

STONE (J. C.) and MILLIS (J. F.). Solid geometry. Chicago, Sanborn, 1916. Pp. 8 + 277–450. $0.85

STRATTON (W. T.) and REMICK (B. L.). Agricultural arithmetic. New York, Macmillan, 1916. 8vo. 10 + 239 pp. $0.50

III. APPLIED MATHEMATICS.

ANNUAIRE pour l'an 1916 publié par le Bureau des Longitudes, contenant des notices par G. Bigourdan et E. Picard. Paris, Gauthier-Villars, 1916. 6 + 657 pp. + 3 cartes magnétiques. Cartonné. Fr. 2.00

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BAZARD (—.). Cours de mécanique. Tome 5: Moteurs à combustion interne. Paris, Geisler, 1916. 8vo. 382 pp. Fr. 12.00

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BIGOURDAN (G.). See ANNUAIRE.

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FLAMARD (E.). Etude sur les méthodes nouvelles de la statique des constructions. Paris, Gauthier-Villars, 1914. 6 + 114 pp. Fr. 5.00

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VECELLIO (A.). Equilibrio cosmico. Parte I: L'equilibrio nel mondo fisico. Feltre, tip. P. Castaldi, di O. Boschiero, 1916. 8vo. 89 pp. L. 1.00

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Book No
L R RY

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-sixth regular meeting of the Society was held in New York City on Saturday, October 28, 1916. The attendance at the morning and afternoon sessions included the following forty-two members:

Professor M. J. Babb, Dr. F. W. Beal, Mr. D. R. Belcher, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Mr. G. W. Evans, Dr. C. A. Fischer, Professor T. S. Fiske, Dr. C. H. Forsyth, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor C. C. Grove, Professor M. W. Haskell, Professor H. E. Hawkes, Professor Dunham Jackson, Dr. L. L. Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. R. Kline, Mr. E. H. Koch, Jr., Mr. Harry Langman, Professor H. B. Mitchell, Professor R. L. Moore, Mr. G. W. Mullins, Professor H. W. Reddick, Mr. J. F. Ritt, Mrs. J. R. Roe, Dr. Caroline Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor Oswald Veblen, Dr. J. H. Weaver, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. A. C. Bose, Calcutta, India; Professor L. C. Emmons, Michigan Agricultural College; Professor A. M. Harding, University of Arkansas; Dr. W. L. Hart, Harvard University; Dr. J. R. Musselman, University of Illinois; Mr. S. Z. Rothschild, Sun Life Insurance Company, Baltimore, Md.; Professor Pauline Sperry, Smith College. Six applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council to be elected at the annual meeting. Committees were appointed to arrange for a joint session at the annual meeting with other scientific bodies and for the summer meeting of 1917.

The following papers were read at the October meeting:

(1) Mrs. J. R. ROE: "Interfunctional expressibility problems of symmetric functions."

(2) Professor E. D. ROE, JR.: "A geometric representation."

(3) Professor E. D. ROE, JR.: "Studies of the Kreisteilungsgleichung and related equations."

(4) Professor E. D. ROE, JR.: "The irreducible factors of $x^{n-1} + x^{n-2} + \cdots + 1$."

(5) Professor H. B. MITCHELL: "On the imaginary roots of a polynomial and the real roots of its derivative."

(6) Dr. J. H. WEAVER: "Some properties of parabolas generated by straight lines and circles."

(7) Professor F. N. COLE: "Complete census of the triad systems in fifteen letters."

(8) Professor O. E. GLENN: "Translation surfaces associated with line congruences."

(9) Professor O. E. GLENN: "Methods in the invariant theory of special groups, based on finite expansions of forms."

(10) Professor R. L. MOORE: "A theorem concerning continuous curves."

(11) Dr. J. R. KLINE: "The converse of the theorem concerning the division of a plane by an open curve."

(12) Mr. H. S. VANDIVER: "Note on the distribution of quadratic and higher power residues."

(13) Mr. H. S. VANDIVER: "The generalized Lagrange indeterminate congruence for a composite ideal modulus."

Professor Roe's first two papers were read by Mrs. Roe; his third paper, Professor Glenn's first paper, and Mr. Vandiver's papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If α is a complex of numbers, b the elementary symmetric functions of the α's, S the power sums, and t the complete symmetric functions, then, of the complex α, Mrs. Roe undertakes to investigate the relations of the four kinds of functions, viz.: the monomial symmetric functions, the power sums, the elementary symmetric functions and the complete symmetric functions. If the four kinds of functions are taken two at a time, six combinations arise, and a consideration of the expressibility of each of a pair in terms of the other leads to a systematic study of twelve tables of coefficients.

The principal object of the investigation is to derive the properties of, and interfunctional relations and general formulas for, the coefficients in these tables.

The problems of the t's and the α's and also those of the S's and the α's are especially considered in this paper. Tables up to weight ten illustrate the results obtained.

2. Professor Roe adjoins the complex plane to the x axis of the real plane for the graphical realization of the complex values, when such exist, of y in $y = f(x)$, and also for the purpose of representing some real curves in space by a single equation in x and y. The complex plane is adjoined so that its origin is always in the axis of x with its axis of positive reals parallel to the positive direction of the axis of y, in fact in the real plane, as the complex plane slides along always perpendicular to the x axis at the distance x as x changes. By this representation every curve has an actual locus from $-\infty$ to $+\infty$. The method is applied to the function

$$y = \{x^{x-1}(x-1)^x\}^{1/(1-2x)}.$$

In the usual simplest real representation this consists of a curve in the real plane between $x = -\infty$ and $x = 0$, and $x = 1$ to $x = +\infty$, but between $x = 0$ and $x = 1$ consists only of discrete points. But if the complex value of y corresponding to a value of x is laid off on the complex plane adjoined to the x axis at the distance x, the curve between 0 and 1 becomes multiple valued for each simplest branch, continuous, and consists of an infinite number of spirals which pierce the real plane in the discrete points and all lie on a real surface of revolution $y^2 + z^2 = [\text{mod } \{x^{x-1}(x-1)^x\}^{1/(1-2x)}]^2$ where x, y, z are the coordinates of real space of three dimensions. Finally the general equation in x and y alone of all real spirals in space of three dimensions winding around the axis of x is shown to be

$$y = f_1(x)e^{f_2(x)i}.$$

3. In this paper Professor Roe studies the four equations $\cos n\varphi \pm \cos (n+1)\varphi = 0$, $\cos (n-1)\varphi \pm \cos (n+1)\varphi = 0$, obtaining the number of and expressions for their distinct roots and applying the results to some geometrical constructions. The "Kreisteilungsgleichung" in algebraic form (derivable from the second equation) is studied and an explicit expression is obtained for the general coefficient of any term in it. The same is also done for a like algebraic equation derivable from the first equation above.

4. In this paper Professor Roe considers the irreducible factors of $f(x, n) \equiv x^{n-1} + x^{n-2} + \cdots + 1$ by means of sums of powers of the roots of $(x - 1)f(x, n) = 0$.

5. The object of Professor Mitchell's paper is the deduction of simple criteria which may enable us to infer, without laborious calculation, the approximate position of a pair of imaginary roots of a polynomial with real coefficients, when the existence of imaginary roots is revealed through the vanishing of the derivative, (*a*) beyond the interval between the greatest and the least real root of the polynomial, (*b*) more than once in an interval between two consecutive real roots. Under the first assumption, upper and lower limits are determined which depend only upon the degree of the polynomial and the distance between a real root of the derivative and the adjacent real root of the polynomial. Under the second assumption it is shown that if the polynomial have not more than 8 imaginary roots, the real part of at least one pair must lie in the interval where the derivative vanishes more than once, and that the coefficient of i must be less in absolute value than the extent of that interval. When the polynomial has more than 8 imaginary roots, but when the number of real roots on one side of the interval is not greater than 8, it is shown that not all of the imaginary roots can have their real parts without and on one side of the interval —this restriction thus applying to all polynomials with real coefficients whose degree is less than 20.

The criteria are particularly suited to graphical analysis, as they involve only the approximate positions of the real roots and bend points.

6. Starting with the well-known fact that the locus of the center of a circle tangent to a given straight line and a given circle is two parabolas (one of which may be degenerate), Dr. Weaver proves several theorems showing relations between the three curves and involving questions of collinearity and concurrence. He then extends some of these results to the triangle and its circles and sets forth some sets of perspective triangles for which the axis of perspectivity passes through the center of perspectivity.

7. Professor H. S. White has determined all the triad sys-

tems in fifteen letters that have groups of substitutions into themselves, the number of types being 44.* Professors White and Cummings have also identified 33 groupless systems in fifteen letters, but a complete census of the groupless systems remained to be carried out. No method of doing this is known except that of actual sifting of all possible cases.

In forming triad systems in fifteen letters 1, 2, 3, $\cdots$, 15 the seven triads that contain 1 may be written in the style 1 2 3, 1 4 5, 1 6 7, 1 8 9, 1 10 11, 1 12 13, 1 14 15. For the triads containing 2 there are then four types of opening

(1) 2 4 6, 2 5 7, 2 8 10, 2 9 11, 2 12 14, 2 13 15;
(2) 2 4 6, 2 5 7, 2 8 10, 2 11 12, 2 13 14, 2 9 15;
(3) 2 4 6, 2 7 8, 2 5 9, 2 10 12, 2 13 14, 2 11 15;
(4) 2 4 6, 2 7 8, 2 9 10, 2 11 12, 2 13 14, 2 5 15;

which may be designated as the triple tetrad, single tetrad, hexad, and duodecad types, respectively. Professor Cole has found that the only triad system in fifteen letters that can be made up from hexads and duodecads exclusively is the known system of Heffter. All others involve tetrads.

If to the tetrad 1 2 3, 1 4 5, 1 6 7; 2 4 6, 2 5 7 there be added the triads 3 4 7, 3 5 6, the result is a triad system in seven letters within that of the fifteen letters, which is then said to have a 7-head, or merely head. If 3 5 6 is replaced by 3 5 8, we may perhaps speak of a semi-head. The complete census of the triad systems in fifteen letters may then be tabulated as follows:

I. Types with triple tetrads, 60, of which 22 have heads, 12 have semi-heads but no heads, and 26 have neither head nor semi-head.

II. Types with tetrads but no triple tetrads, 19, of which 1 has a head, 3 have semi-heads but no heads, and 15 have neither head nor semi-head.

III. Heffter's headless system, 1.

Total, 80 systems.

The 23 types with heads all have groups; of the headless types 21 have groups and 36 are groupless.

8. Professor Glenn's first paper appeared in full in the December BULLETIN.

9. A paper by Professor Glenn published in the *Trans-*

* See this BULLETIN, vol. 21, pp. 164, 488.

actions in 1914 contains the determination of the expansion of the general binary form f of order $n(m+1)-1$ as a binary m-ic in two arbitrary forms f_{1n}, f_{2n} of order n, the coefficients of the expansion being quantics φ_{in-1} $(i = 0, \cdots, m)$ of order $n-1$. Assuming, in the present paper, that f_{1n}, f_{2n} are the universal covariants of a definite linear group G, it is shown that the φ_{in-1} are covariants; and a complete system of f under G is the simultaneous system of the set φ_{in-1}, f_{1n}, f_{2n}. Particular cases are treated; the most elementary being an explicit derivation of the (known) system of orthogonal concomitants of an m-ic.

As a result of applying methods similar to the above in the case when G is the group of binary linear transformations modulo p, $p \neq 2$, a superior limit for the order of an irreducible formal covariant modulo p of any binary form or set of forms is determined.

10. Professor Moore proposes to show that every two points of a continuous curve (no matter how crinkly it may be) can be joined by a *simple* continuous arc that lies wholly in the curve.

11. In his paper* "On the foundations of plane analysis situs," Professor R. L. Moore defines an open curve as a closed connected set of points M such that if P is any point of M, then $M-P$ is the sum of two mutually exclusive connected sets of points, neither of which contains a limit point of the other. He shows that if l is an open curve and S is the set of all points, then $S-l=S_1+S_2$, where S_1 and S_2 are connected point sets, such that an arc from a point of S_1 to a point of S_2 contains at least one point of l. This theorem is proved on the basis of his set of axioms Σ_3 and is therefore true in certain spaces which are neither metrical, descriptive nor separable.

Dr. Kline proves the converse of this theorem for open curves. The statement of the converse theorem is as follows:

Suppose K is a closed set of points and that $S-K=S_1+S_2$ where S_1 and S_2 are non-compact point sets such that

(1) every two points of S_i $(i = 1, 2)$ can be joined by an arc lying entirely in S_i.

(2) every arc joining a point of S_1 to a point of S_2 contains a point of K.

* *Transactions*, vol. 17, no. 2, p. 131 (April, 1916).

(3) if O is a point of K and P is any point not belonging to K, then P can be joined to O by an arc having no point except O in common with K.

Every point set K that satisfies these conditions is an open curve.

12. If p is a prime of the form $4n+3$, consider the number of quadratic residues included in the set $1, 2, \cdots, 2n+1$. In the present note Mr. Vandiver proves a theorem which sets forth a connection between this number and the number of quadratic residues in any set defined by $h+[ah]<p$, where a is a fixed integer less than $p-1$ and h ranges over the set $1, 2, \cdots, p-1$, the expression $[ah]$ denoting the least positive residue of ah, modulo p. Analogous theorems are also found concerning the distribution of higher power residues.

13. Consider the indeterminate congruence of Lagrange

$$(x-1)(x-2)\cdots(x-(p-1)) \equiv x^{p-1}-1 \pmod{p},$$

where x is an indeterminate and p is a prime integer. Mr. Vandiver obtains some generalizations of this relation such that the set $1, 2, \cdots, p-1$ modulo p is replaced by all the incongruent residues of a composite ideal modulus which are prime to the modulus. The paper will appear in the *Annals of Mathematics*.

F. N. Cole,
Secretary.

CORRECTION.

The following regretable errata in the reports of the summer meeting and colloquium of the Society, published in the November Bulletin, have been brought to the attention of the Secretary:

I. In the report of the summer meeting, page 65, it is stated that Professor C. N. Moore's paper appeared in full in the October Bulletin. A paper with the same title did appear in the October Bulletin, but it was read at the annual meeting held last January. The abstract of Professor Moore's summer meeting paper is printed below, with apologies to the author.

10. In Professor Moore's paper an example is given of a function continuous in an interval $0 \leqq x \leqq 1$, whose development in Bessel's functions is not summable $(C\lambda)$ at the point $x = 0$, for any value of λ in the interval $0 \leqq \lambda < \frac{1}{2}$.

II. In the report of the colloquium, pages 85–88, Professor Veblen's subject matter appears distributed in six "Lectures," whereas only five lectures were actually delivered by each author. The headings in Professor Veblen's synopsis represent certain divisions of the subject matter, not the division into lectures, and the word "Lecture" should have read "Section."

THE MAXIMUM NUMBER OF CUSPS OF AN ALGEBRAIC PLANE CURVE, AND ENUMERATION OF SELF-DUAL CURVES.

BY PROFESSOR M. W. HASKELL.

(Read before the San Francisco Section of the American Mathematical Society, October 24, 1914.)

It is well known that the double points of a rational algebraic curve can not in general all be cusps, and the maximum number of cusps has been determined in certain special cases. It is not difficult to find the maximum number from the consideration that none of the numbers in Plücker's equations can be negative.

Let m be the order and n the class, d the number of double points, k of cusps, i of inflexions and t of double tangents. We may first assume $d = 0$. In this case

$$2t = [m^2 - 9 - 3k][m^2 - 2m - 3k]$$

and the following inequalities must be satisfied:

$$3k < m(m-1), \tag{1}$$

$$8k \leqq 3m(m-2), \tag{2}$$

$$2k \leqq (m-1)(m-2), \tag{3}$$

(4) Either $3k \leqq m^2 - 9$ and $3k \leqq m^2 - 2m$ simultaneously or else

(5) $3k \geqq m^2 - 9$ and $3k \geqq m^2 - 2m$ simultaneously.

Now, if m is greater than 9, $m^2 - 9$ is greater than $m(m-1)$ while if $m = 7, 8, 9$, $m^2 - 9$ is greater than $\frac{9}{8}m(m-2)$, so that if m is greater than 6, (4) must be satisfied. Hence, if m is greater than 6,

$$k \leqq \tfrac{1}{3}m(m-2).$$

Indeed, this formula is satisfied for $m = 3$ and $m = 5$, as is well known. The only exceptions are therefore $m = 4$ and $m = 6$.

Now, if m is of the form $3k$ or $3k + 2$, it is easy to show that a curve with the maximum number of cusps can have no further double points. The curve is self-dual, and

$$n = m, \quad d = t = 0, \quad k = i = \tfrac{1}{3}m(m-2),$$
$$p = \tfrac{1}{6}(m-2)(m-3).$$

If, however, m is of the form $3k + 1$, the curve may have one double point in addition. This curve is also self-dual, and

$$n = m, \quad d = t = 1, \quad k = i = \tfrac{1}{3}[m(m-2) - 2],$$
$$p = \tfrac{1}{6}(m-1)(m-4).$$

This result allows us to enumerate all self-dual curves. For, if $n = m$,

$$k = i = m - 2 + 2p,$$
$$d = t = \tfrac{1}{2}(m-2)(m-3) - 3p,$$

so that the minimum number of cusps for a self-dual curve is $m - 2$, and there is hence just one self-dual curve for every value of m corresponding to each value of p from zero up to the value given above.

Moreover, since

$$k = m - 2 + 2p \leqq \tfrac{1}{3}m(m-2),$$

the lowest order of a self-dual curve of given deficiency $p(>0)$ is the smallest integer satisfying the inequality

$$m \geqq \frac{5 + \sqrt{24p + 1}}{2}.$$

NEW YORK,
November, 1916.

NOTE ON ASYMPTOTIC EXPRESSIONS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.*

BY PROFESSOR W. E. MILNE.

(Read before the American Mathematical Society, December 28, 1915.)

LET n independent solutions of the linear differential equation

$$(1) \qquad \frac{d^n u}{dx^n} + P_2(x)\frac{d^{n-2}u}{dx^{n-2}} + \cdots + P_n(x)u + \rho^n u = 0$$

be denoted by $y_1, y_2, \cdots, y_n$. It is the aim of this note to establish asymptotic representations of a particular form for the n functions $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ determined by the n identities

$$(2) \qquad \sum_{i=1}^{n} y_i^{(s-1)}\bar{y}_i = \begin{cases} 0, \text{ if } s = 1, 2, \cdots, n-1, \\ 1, \text{ if } s = n. \end{cases}$$

For this purpose we employ asymptotic forms for the y's, as follows.† If the coefficients $P_s(x)$ in (1) have continuous derivatives of order $(m + n - s)$, m being a positive integer or zero, in the interval $a \leqq x \leqq b$, then there exist n independent solutions of (1) of the form

$$(3) \qquad \begin{aligned} y_i &= u_i(x, \rho) + e^{\rho w_i(x-c)}E_{i0}/\rho^{m+1}, \\ y_i^{(k)} &= u_i^{(k)}(x, \rho) + e^{\rho w_i(x-c)}E_{ik}/\rho^{m+1-k} \\ &\qquad (i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, n-1), \end{aligned}$$

where

$$(4) \qquad u_i(x, \rho) = e^{\rho w_i(x-c)}\left[1 + \frac{\varphi_1(x)}{\rho w_i} + \cdots + \frac{\varphi_m(x)}{(\rho w_i)^m}\right].$$

The functions $\varphi_j(x)$ have continuous $(m + n - j)$th derivatives in (a, b) and are independent of i, while for x in (a, b)

* The formulas given here were published without proof in the *Proceedings Nat. Academy of Sciences*, vol. 2 (1916), pp. 543–5.

† The existence of asymptotic solutions of (1) in nearly the form given in (3) was proved by Birkhoff, *Transactions Amer. Math. Society*, vol. 9 (1908), pp. 219–231, 381–2. The proof of the formulas in (3) and (4) is conducted in a similar manner and offers no essentially new difficulty. For an explicit statement of the difference between Birkhoff's formulas and those here given, see the note in the *Proceedings* referred to above.

and ρ in a suitably chosen sector* S_k of the ρ-plane the E_{ij} are analytic in ρ and bounded as ρ becomes infinite.

By the use of the above formulas we can now show that *when the coefficients $P_s(x)$ have continuous $(m+n-s)$th derivatives in (a, b), the n functions $\bar{y}_i$ have when $|\rho|$ is large the asymptotic form*†

$$\bar{y}_i = \frac{e^{-\rho w_i(x-c)}}{n(\rho w_i)^{n-1}}[\bar{v}_i(x, \rho) + \bar{E}_i/\rho^{m+1}] \qquad (i = 1, 2, \cdots, n), \tag{5}$$

where

$$\bar{v}_i(x, \rho) = 1 + \psi_1(x)/\rho w_i + \cdots + \psi_m(x)/(\rho w_i)^m. \tag{6}$$

The functions $\psi_j(x)$ are independent of i and have continuous $(m+n-j)$th derivatives in (a, b), and for x in (a, b) and ρ on S_k the $\bar{E}_i$ are analytic in ρ and bounded as ρ becomes infinite.

The method of proof is to substitute in (2) from (5) and (6) and show that the m ψ's and n $\bar{E}$'s can be chosen to satisfy (2). It will readily be seen that $y_i^{(k)}$ can be written

$$y_i^{(k)} = (\rho w_i)^k e^{\rho w_i(x-c)}[v_{ik}(x, \rho) + E_{ik}'/\rho^{m+1}], \qquad (i = 1, 2, \cdots, n;\ k = 0, 1, \cdots, n-1), \tag{7}$$

where

$$v_{ik}(x, \rho) = 1 + \varphi_{1k}/\rho w_i + \cdots + \varphi_{mk}/(\rho w_i)^m, \tag{8}$$

and

$$\varphi_{jk} = \sum_{r+t=j}\left[\frac{k!}{r!(k-r)!}\right]\varphi_t^{(r)}.$$

Substitute in (2) from (5) and (7), cancel the exponentials, and divide out the factor $-1/n\rho^{n-s}$ from the sth equation. The result is ‡

$$\sum_{i=1}^{n} w_i^s v_{i,s-1}\bar{v}_i + \frac{1}{\rho^{m+1}}\left[\sum_{i=1}^{n}(w_i^s + E/\rho)\bar{E}_i + E\right] = \begin{cases} 0, \text{ if } s = 1, 2, \cdots, n-1, \\ -n, \text{ if } s = n. \end{cases} \tag{9}$$

* For the precise definition of S_k see either Birkhoff, loc. cit., or the note in the *Proceedings*.

† A similar but less explicit formula was used by Birkhoff, loc. cit., p. 391, formula (56).

‡ The letter E is here used in a generic sense to denote any function analytic in S_k and bounded in S_k as ρ becomes infinite.

By (6) and (8)

$$(10)\qquad \sum_{i=1}^{n} w_i^s v_{i,\,s-1}\bar{v}_i = \sum_{i=1}^{n}\sum_{j=0}^{m} \theta_{js}\rho^{-j} w_i^{s-j} + E/\rho^{m+1},$$

where

$$\theta_{js} = \sum_{r+t=j} \varphi_{r,\,s-1}\psi_t.$$

Now $\sum_{i=1}^{n} w_i^{s-j}$ is zero except when $j - s = kn$, where k is integral, in which case it is $(-1)^k n$. Therefore, if we denote $kn + s$ by z, the double sum in (10) reduces to

$$\sum_{k=0}^{I_s} (-1)^k n\theta_{zs}/\rho^z$$

when $s = 1, 2, \cdots, n-1$, and to

$$-n + \sum_{k=0}^{I_s} (-1)^k n\theta_{zs}/\rho^z$$

when $s = n$. The symbol I_s denotes the integral part of $(m - s)/n$. Substituting back into (9), we now get

$$\sum_{k=0}^{I_s} (-1)^k n\theta_{zs}/\rho^z + \frac{1}{\rho^{m+1}}\left[\sum_{i=1}^{n} (w_i^s + E/\rho)\bar{E}_i + E\right] = 0$$
$$(s = 1, 2, \cdots, n).$$

Since these are identities in ρ, the coefficient of every power of $1/\rho$ up to and including the mth must vanish separately, giving the m equations

$$(11)\qquad \theta_{zs} = 0, \quad \text{or} \quad \psi_z = -\sum_{r+t=z}^{t<z} \varphi_{r,\,s-1}\psi_t$$
$$(k = 0, 1, \cdots, I_s;\ s = 1, 2, \cdots, n),$$

and leaving the n equations

$$(12)\qquad \sum_{i=1}^{n} (w_i^s + E/\rho)\bar{E}_i + E = 0 \quad (s = 1, 2, \cdots, n).$$

From equations (11) the m functions ψ_j may be determined in succession, and are readily shown by mathematical induction to have continuous derivatives of order $(m + n - j)$ in (a, b). The functions represented by the letter E in equations (12) can be expressed in terms of the φ's, the ψ's, and the E_{ik}'s,

and are therefore now to be regarded as known functions. When $|\rho|$ is sufficiently large the determinant of the coefficients in (12) is not zero, so the $\overline{E}_i$'s can be uniquely determined. It is plain that for ρ in S_k they are analytic in ρ and bounded as ρ becomes infinite.

BOWDOIN COLLEGE,
October, 1916.

ON NOTATIONAL EQUIVALENCE.

BY PROFESSOR EDWIN BIDWELL WILSON.

IN reply to my query* to Dr. Poor "Why not make the work short?" he states† that brevity was not his aim, that one of his purposes was to exhibit the Burali-Forti and Marcolongo notation. I must accept that answer and admit my error in assuming that his only aim was to derive as directly as possible some transformations which are needed in certain studies in applied mathematics. It is, however, difficult for me to admit many of his other contentions. I have no desire to enter upon any polemic in regard to these matters, but it does seem that further explanation from Dr. Poor would be valuable to all who are interested in vectorial methods.

1. He states: That the use of words, such as grad, div, rot, is hampering seems to be a matter of opinion, since they may be used interchangeably with other symbols.

I hold that because two sets of symbols may be used as interchangeably as these and ∇ is no criterion at all that one is not more hampering than another. For instance, 94 and XCIV are equivalent symbols, so are 8 and VIII, and also 752 and DCCLII. Yet for the arithmetical operation of multiplying eight and ninety-four the Arabic notation is far superior to the Roman (or Greek); indeed so marked is the superiority that one may well wonder how far mathematics would now be advanced had no better system than the Roman been devised.

May we not fairly maintain that notationally Arabic and Roman numerals are not interchangeable? Is it true that two notations in terms of which premises and conclusions may both be stated are for that reason interchangeable? To

* Wilson, this BULLETIN, vol. 22, April, 1916, p. 336.
† Poor, this BULLETIN, vol. 22, July, 1916, p. 503.

me the whole series of formal operations by means of which the passage is made from premise to conclusion must be given a fundamental place in determining the question of notational interchangeability in so far as concerns the discussion of what is hampering and what felicitous.

What was Maschke's object in developing a symbolic method, analogous to that of Clebsch-Aronhold, for dealing with differential forms? It could hardly have been merely to introduce a new notation interchangeable with older ones with all the attendant liability to irritation and confusion.

2. Dr. Poor states that Burali-Forti and Marcolongo have pointed out how the dyadics of Gibbs constantly depend on cartesian coordinates, a non-linear system.

Dr. Alexander Macfarlane called my attention to this remark (and others) in one of his letters written shortly before his death. He did not, nor do I, believe all the things those eminent authors say about Gibbs' system—or else I do not understand them. If it is meant that in the Gibbs-Wilson Vector Analysis many properties of dyadics are proved by means of cartesian coordinates, I should admit that some proofs of that sort were given, but should call attention to a subsequent work* also done after Gibbs' lectures, in which no such proofs are given. I have always taught my students, as I learned from Gibbs, that dyadics are in no way dependent on coordinates, no matter how proofs may be given. I am sorry if I have been propagating heterodoxies.

3. He states: It is unfortunate that Professor Wilson introduced cartesian coordinates into his proof since a coordinate system has no place in vector analysis.

To be sure, puristic ideals have enraptured the gaze of some students of vector analysis, as of some students of projective geometry and other branches of mathematics. The refinements of method which have sometimes resulted from this striving toward the puristic goal have been of great value. Whether, however, the insistence on such ideals under all circumstances is fortunate is a mooted question. Certainly it is possible to define cartesian coordinates in terms of vectors (a method recommended by Heaviside); it is then the coordinates which depend upon vector analysis and consequently have as much place in it as anything else.

* Wilson, *Trans. Conn. Acad. Arts Sci.*, New Haven, vol. 14, 1908, pp. 1–57.

The case in which I used cartesian coordinates in my note was in proving the operational identity $\int dS() = -\int d\tau()$. The proof of this well-known formula was not essential to my work—I could have merely quoted and applied the result as Dr. Poor quoted and applied many results from Burali-Forti and Marcolongo.*

4. He states: The notation of Burali-Forti and Marcolongo could have been made to compare very favorably with Professor Wilson's compact reproduction of the formulas in the Gibbs notation.

It would please me much to see this done. It is suggested that we need only compare the analytic statement of the theorems in the two notations. I am unable to make the comparison satisfactorily, or to admit that the comparison if made would be a just criterion. To my mind it would be necessary to compare the whole proofs in detail, and in particular to determine what propositions were needed for the proofs in the two notations which were not obvious consequences of the simple laws of operation in the respective notations.

For instance, Dr. Poor writes the relation†

$$\mathrm{div}_M \alpha \boldsymbol{u} = -\boldsymbol{u} \times \mathrm{grad}_P K\alpha,$$

$\boldsymbol{u}$ independent of M, which in Gibbs' notation is

$$\nabla_M \cdot (\alpha \cdot \boldsymbol{u}) = (\nabla_M \cdot \alpha) \cdot \boldsymbol{u} = -\boldsymbol{u} \cdot (\nabla_P \cdot \alpha),$$

and is obvious (it being understood that $\nabla_M = -\nabla_P$). But the relation is derived by Dr. Poor from a quoted formula

$$\mathrm{div}_M \alpha \boldsymbol{u} = \boldsymbol{u} \times \mathrm{grad}_M K\alpha + I_1\left(\alpha \frac{d\boldsymbol{u}}{dM}\right)$$

from Burali-Forti and Marcolongo. It may be that there is some way of remembering all such formulas in their notation, but I have never deciphered any. In the Gibbs notation we should have

$$\nabla \cdot (\alpha \cdot \boldsymbol{u}) = (\nabla \cdot \alpha) \cdot \boldsymbol{u} + \nabla_1 \cdot (\alpha \cdot \boldsymbol{u}_1) = (\nabla \cdot \alpha) \cdot \boldsymbol{u} + \nabla \boldsymbol{u} : \alpha$$

by the usual method of differentiation in situ.

* There seem to be ten such citations. See, Poor, this BULLETIN, vol. 22, January, 1916, pp. 174–181.

† Loc. cit. above, p. 178.

The important thing here is fundamental to the whole question of notation and particularly to notational interchangeability. The rule of differentiation in situ and the ordinary rules for the use of dot and cross in vector algebra taken with the identity $\int dS() = -\int d\tau()$ suffice to prove all Dr. Poor's theorems and many others of the sort without reference to any list of formulas—the whole thing has become mere formal operation which for a student of Hamilton, Tait, Gibbs, and McAulay is in the same category as the work

$$a - \frac{1}{a} = \frac{a^2 - 1}{a} = \frac{(a+1)(a-1)}{a}$$

is for the schoolboy.* If this is equally true of the student of Burali-Forti and Marcolongo, I am both surprised and happy.

ON PIERPONT'S INTEGRAL. REPLY TO PROFESSOR PIERPONT.

BY PROFESSOR MAURICE FRÉCHET.

My single aim in my previous contribution to this journal ("On Pierpont's definition of integrals," volume 22, number 6, March, 1916) was to point out that, in my own words, *this new definition is inappropriate. I still hold to my original assertion* (though for partly different reasons) and will show why I do so.

Thus the question whether two non-measurable sets with no points in common are separated or not is far from being the vital point. This being explicitly stated, I hasten to say that *concerning this last particular question,* Professor Pierpont is entirely justified in saying: "Professor Fréchet has been misled at this point . . . and his example establishes not an error on my part but a carelessness of reasoning on his." As a matter of fact, I too quickly assimilated in my mind "separated" with "having no point in common." The same thing occurred with the word "exterior" and my objection to theorem 341, page 346 arose from a miscon-

* It would not have been obvious to the schoolboy, perhaps not even to a professional mathematician, in the days before a suitable notation for elementary algebra had been developed.

ception of the meaning of this word in Pierpont's terminology. Indeed, as I was at the front (where I am still) when I wrote this article, I could only compose it out of notes formerly taken. And even now, I do not understand what Professor Pierpont describes as "sets exterior to each other." I will then let fall my objections to formulas (2), (5), (7), in Professor Pierpont's reply.

However, under present circumstances (I write this on June 30) it is wiser not to postpone my answer. And with but the information I have in hand, I will make good my point as follows:

The main differences between Lebesgue and Pierpont integrals are two in number.

I. When E is measurable and $f(x)$ is summable the common value of Lebesgue and Pierpont integrals $\int_E f(x)dx$ is arrived at in different manners.

Then, I still maintain that in this case the real difference between their definitions is *not* that—as Professor Pierpont asserts—he makes use of an infinite instead of a finite number of parts δ_i of E (as in Riemann's definition). It lies essentially in the use of measurable sets instead of intervals. For, in the most important case: when $f(x)$ is bounded, the finiteness or infiniteness of the number of parts δ_i of E is indifferent in Pierpont's definition. This is easily seen, starting from Pierpont's theorem, that the remainder

$$\overline{\text{meas.}}\ \delta_i + \overline{\text{meas.}}\ \delta_{i+1} + \cdots$$

(of a series equal to $\overline{\text{meas.}}\ E$) converges to zero.

It is open to Professor Pierpont to prove that this former assertion of mine is wrong.

I mention in passing that apropos of a different memoir, M. Lebesgue kindly pointed out to me that a definition of Lebesgue's integrals by means of Riemann's sums was given as early as 1905 by W. H. Young (*Philosophical Transactions*).

II. Pierpont's definition enables him to give a definite value to integrals which are not integrable according to M. Lebesgue.

Whereas when E is non-measurable, M. Lebesgue contents himself with saying that the measure of E is contained between $\overline{\text{meas.}}\ E$ and $\underline{\text{meas.}}\ E$, Professor Pierpont goes further. According to his definition, the function $f(x) \equiv 1$ is integrable

on any set E and

$$\int_E dx = \overline{\text{meas.}}\ E.$$

It is questionable *whether this precise formula is a decided improvement over M. Lebesgue's statement.* But, further, from this formula it is deduced that *the Pierpont integral does not enjoy the fundamental property that if E, F are sets with no points in common*

$$\int_{E+F} f(x)dx = \int_E f(x)dx + \int_F f(x)dx$$

(which however is true when E, F are "separated," according to Professor Pierpont). It suffices to apply this formula when $f(x) \equiv 1$, $E + F$ is an interval and E is non-measurable.

A REPLY TO A REPLY.

BY PROFESSOR JAMES PIERPONT.

As I view the issue between Professor Fréchet and myself, it may be summed up as follows:

1°. Professor Fréchet thought that it was possible to split a measurable set into two *separated* non-measurable sets, and he gave an alleged example. Since no such division is possible this example proved to be an ignis fatuus.

2°. Supported by this example, it was easy for Professor Fréchet to bring a number of grave charges against my work, in fact it might seem as if my whole theory had toppled to the ground.

3°. Professor Fréchet now admits (provisionally) that he was in error on this score, but he still holds to his "original assertion" that my integral definition "is inappropriate," "though for partly different reasons." What are these new reasons? Although I have read and reread the above article I have found but one, viz.: Suppose A is *non*-measurable and suppose B and C form a *non*-separated division of A, then the relation

$$\int_A = \int_B + \int_C \tag{1}$$

may not hold.

This very obvious fact I have known from the start; it is one that any one would discover. To Professor Fréchet this may be an insuperable objection and I have no contention with any one who holds this view. A similar peculiarity is presented in many theories. For example, one may take the stand that the double series

$$
\begin{aligned}
& a_{11} + a_{12} + a_{13} + \cdots \\
(2) \qquad & + a_{21} + a_{22} + \cdots \\
& + a_{31} + \cdots \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

is convergent only when it is absolutely convergent,* but many authorities do not. In the latter case one is led to a number of unexpected results; e. g., the series (2) may converge although every series formed of a row or a column of (2) is divergent.

To my mind it does not seem wise to be doctrinaire in such matters. The relation (1) does hold for separated divisions of A, and when A is measurable no other divisions are possible. Since no one as yet has exhibited a non-measurable set, only the existence of such sets having been established, it seems at least premature to argue on a priori grounds against any theory which makes a step in advance.

In any case the nature of Professor Fréchet's objections has been widely changed; as first formulated they struck at the very foundation of my theory by impeaching the correctness of one of its main theorems; at present the only objection I see is an expression of a personal opinion.

4°. Polemics are apt to be interminable; fresh charges are made, fresh rejoinders necessitated and so on ad infinitum. I therefore am not astonished that Professor Fréchet has injected a new element into the discussion. It now seems that the vital point is the "real difference" between Lebesgue's definition and my own. To me this question is one of complete indifference and I leave Professor Fréchet to settle it entirely to his own satisfaction.

* Cf. C. Jordan, Cours d'Analyse, vol. I, p. 302.

SHORTER NOTICES.

The Teaching of Algebra (including Trigonometry). By T. PERCY NUNN, M.A., D.Sc. London, Longmans, Green and Company, 1914. xvi + 616 pp. Price $2.

Exercises in Algebra (including Trigonometry). By T. PERCY NUNN. Part I. 1913. xii+422 pp. Price $1.10. Part II. 1914. xii+552 pp. Price $1.75.

A FEW years ago one would have been justified in expressing surprise had an American teacher of mathematics gone to England to study modern tendencies in secondary education. To-day, however, if one were to study the present efforts at reform in the teaching of mathematics, he would certainly go to England for some of the most prominent leaders in the movement. Fifteen years have wrought a great change, and while the change has not been along the exact lines suggested by Professor Perry there is no doubt that he was the most influential of those who awakened the teachers of mathematics in the English-speaking world from the lethargy into which they had fallen. Naturally, the movement to better the work in mathematics has not been uniformly successful; for many ill-considered attempts have been made in addition to the few which give reasonable promise of producing satisfactory results. England has had her extremists as America has had hers, but on the whole she has been more consistent than we in her insistence upon a high degree of scholarship.

Among the strongest advocates both for reform in teaching and for a maintenance of high scholarship is Dr. Nunn, vice-principal of the London Day Training College, a teacher of experience and a man who adds character to the title of educator. In the publication of the three works under review, Dr. Nunn has doubtless been assisted by the judgment of the editors of the series of which these books are a part, Messrs. Abbott and Jackson and Dr. Macaulay, three of the best-known leaders in the progress of secondary mathematics in England.

The general introduction considers the nature of algebra, the formula, the graph; and the questions of method and the curriculum. Part I first discusses non-directed numbers, then directed numbers, and finally logarithms. Part II is concerned chiefly with the trigonometry of the sphere (in

particular, projections), complex numbers, periodic functions, limits, and statistics. Each of these topics is fully amplified by the accompanying exercise manuals where abundant material is given for the student's mastery of the respective chapters.

To the American teacher it will be of chief interest, so far as a review is concerned, to have attention called to a few of those features of the books which reveal the new and richer field which it is proposed to open to the English student. One of the advantages of so doing is that we Americans may see a serious effort at improving algebra without making the subject so soft and insipid as to have no interest for any healthy student. If the reader shall conclude that the teaching of algebra, as here treated, means merely the teaching of Dr. Nunn's algebra, this will not militate against the fact that the book will be found suggestive and helpful to teachers who follow courses which are quite at variance with the one which the author here lays down.

In the first place the reader will be struck by the fact that Dr. Nunn is not merely a mathematician but a philosopher and a psychologist as well. A single excerpt will reveal this fact, but it may be added that the fact is commonly apparent as one reads the pages of the work. "Mathematical truths always have two sides or aspects. With the one they face and have contact with the world of outer realities lying in time and space. With the other they face and have relations with one another. . . . From its dim beginnings by the Euphrates and the Nile mathematics has been on the one hand a means by which man has constantly increased his understanding of his environment and his power of manipulating it, and on the other hand a body of pure ideas, slowly growing and consolidating into a noble rational structure. . . . Our purpose in teaching mathematics in school should be to enable the pupil to realize, at least in an elementary way, this twofold significance of mathematical progress. A person, to be really 'educated,' should have been taught the importance of mathematics as an instrument of material conquests and of social organization, and should be able to appreciate the value and significance of an ordered system of mathematical ideas. There is no need to add that mathematical instruction should also aim at 'disciplining his mind' or giving him 'mental training.' So far as the ideals intended by these phrases are

sound they are comprehended in the wider purpose already stated. . . . The theoretical questions which are of most importance in an elementary course are just those which arise naturally out of attempts to apply mathematical ideas and methods to practical purposes."

With this principle to guide him, Dr. Nunn proposes to consider those parts of algebra which have the highest practical value, and if at times he seems to depart from the path he lays out it should be said that this is with the definite intent to develop his theoretical discussions from a consideration of the practical. In America we have the pedantic term "motivation" to designate this method of approach to the theoretical, but it adds nothing to the idea.

The author's idea is the sound one that the point of approach to algebra is the formula, "illustrations being drawn largely from the 'Pocket Book' of the engineer and similar formularies. . . . A little later comes the study of those manipulations of a formula by which it may be made to yield truths unknown or unperceived before. . . . The gradual elaboration of the formula as an instrument of description and investigation is, then, the first business of the course in algebra."

To carry out this idea the author proposes a fusion plan which has distinct advantages, differing herein from the ill-considered ideas occasionally met of fusing such subjects as algebra and geometry. His idea is that trigonometry, being largely a science of formulas and identities, should grow out of algebra, and that the calculus should do the same. These ideas seem practical, although the details of their execution may vary under such conditions as our colleges impose upon the secondary schools. Carrying out his plan, the author not only introduces work in plane trigonometry, but proceeds to the trigonometry of the sphere, Mercator and great-circle sailing, and map projections. His study of complex numbers develops into a study of circular functions, this into periodic functions, this into progressive and stationary wave motion, and this into hyperbolic functions. The course ends with a very satisfactory presentation of limits, including the elements of differentiation and integration, and with a helpful introduction to the mathematics of statistics.

In the work in graphs Dr. Nunn takes a very sane stand. The extremes of a few years ago find no place in his plan. He recognizes the value of the graph, and gives it a large place

in his scheme, with many examples to show its use; but he also recognizes that "it is inferior to the symbolic formula in many important respects. Its accuracy depends largely upon mechanical or non-intellectual conditions, such as the skill of the draughtsman and the exactness of the squared paper. It is less compact and less easily reproduced. Its message is frequently inarticulate and obscure. For these and similar reasons it should be regarded as a subsidiary algebraic instrument which fulfills its best office when it either leads up to a formula by which it may itself be superseded, or serves to unfold more fully the implications of a formula whose properties have been only partially explored."

After such a carefully considered statement the reader will probably be surprised that Dr. Nunn should resort to graphic treatment in certain cases where it is, to say the least, of doubtful value. Such cases are his elaborate development of the identities $ac + bc = (a + b)c$, $ac - bc = (a - b)c$, and especially $a^2 - b^2 = (a + b)(a - b)$, and the colored graphs in connection with negative numbers.

Another feature in Dr. Nunn's scheme that will strike the reader as of very doubtful value is his postponement of the negative number. Under the older plan of teaching, this delay would have been desirable; but the facility with which the negative number is now illustrated, and the ease with which pupils grasp the idea, renders such delay unnecessary. If the theory is to be presented as scientifically as the author proposes, there is some reason for waiting, but it is probable that the loss exceeds the gain at this stage of algebra.

It is not desirable in a review of this nature to consider further the large number of details that are found in three volumes aggregating over 1,500 pages. The present review has already extended beyond the limits usually allowed in the BULLETIN, but it is proper to add that there has never before appeared a work so helpful to a teacher of algebra. The reviewer does not believe in the sequence that Dr. Nunn lays down; he does not believe in certain uses of the graph that are set forth; he does not believe in several of the efforts at "motivation" which are advanced; and he does not believe that we are ready, in this country at any rate, to undertake certain parts of the course which is planned. But on the other hand he has not failed to recommend the work to hundreds of teachers of mathematics as the most noteworthy

contribution to the teaching of algebra that has yet appeared. Teachers who know our American problem, who are well enough balanced not to be enticed into fields that are certain to resist cultivation at the present time in this country, and who are searching for sane methods of reform, should read the pages which Dr. Nunn has here written with such care and erudition, and with such force and clearness.

DAVID EUGENE SMITH.

Syllabus of Mathematics. A symposium compiled by the Committee on the Teaching of Mathematics to Students of Engineering. Published by the Society for the Promotion of Engineering Education, Ithaca, N. Y., 1912. 136 pp.

THE purpose of this syllabus is to collect those principles and methods of mathematics which should constitute the minimum mathematical equipment of the student of engineering; or "those things for which a student ought never to be obliged to refer to any book—the things which he should have constantly at his fingers' ends." The book contains separate syllabi on elementary algebra (14 pages), elementary geometry and mensuration (7 pages), plane trigonometry (19 pages), analytic geometry (28 pages), differential and integral calculus (44 pages), and complex quantities (3 pages) together with a report of the discussion of these syllabi at the Pittsburgh meeting of the Society (14 pages).

Such syllabi will be of great value to students or teachers who wish to review the essentials of elementary mathematics courses. They will especially aid those students who end courses without proper perspectives. However, some may see danger of low standards in so much stress upon the *minimum* mathematical equipment of an engineer. The ideal of the committee is hardly realized, for few engineers "know by heart" and never need to "look up in a book" all the material in these syllabi. While admitting the value of such synopses of minimum essentials, yet as pointed out in one discussion (page 126), there is also need for lists of *all* the topics and principles that should be included in mathematics courses for engineers since there is some danger of too little rather than too great mathematical equipment.

Only few illustrative problems appear. A valuable supplement would be two sets of problems; one set giving an illustrative problem corresponding to each principle; the

other set giving problems without specifying the particular principles involved. The committee is firmly of the opinion that the principal part of the engineering mathematics course should be problems worked by the students.

When the important principles of elementary algebra may be summed up in fourteen pages, it is surprising to find such universal weakness in algebra among our college students. In addition to the usual topics of algebra are found logarithms, inequalities, and the progressions. The part on exponents would be clearer to students if accompanied by translated word forms, for the full meaning of symbolic statements is often missed. Under neither algebra nor trigonometry is the distinction between equations and identities given. The operations which introduce or take out roots of an equation are not sufficiently brought out. An ingenious form for solving simultaneous linear equations is given.

Under geometry and mensuration we find properties of right and oblique triangles, angles in a circle, proportion of similar figures, area, volume and surface formulas, Cavalieri's theorem, theorems of Guldin or Pappus, and the prismoidal formula all given in seven pages.

The usual topics in trigonometry are given with commendable emphasis placed upon the sine, cosine, and tangent, thus making the other functions secondary. Line functions and graphic methods of remembering certain relations are prominent. The form of generalizing the definitions of the functions for any angle is to be commended.

Most of the part on analytic geometry is devoted to the conic sections. There is not sufficient emphasis placed upon methods of finding the equation when the locus is defined. Under neither algebra nor analytic geometry do we find sufficient treatment given to plotting and locating roots of numerical equations of higher degree; such theory as that usually given under the topic "theory of equations" in college algebra seems entirely missing. The part given to surfaces of the second degree is too condensed to be worth much. The part on the general equation of the second degree might well be condensed or even omitted in a syllabus giving only that which one "should have constantly at his fingers' ends."

The first section on the calculus is a good summary of functions and their graphic representation. In the definition of

function it would be better to emphasize the notion of *correspondence* rather than dependence. Graphs of the elementary functions are given and a short treatment of fitting functions to empirical data is included. In many places the notion of a limit is used but nowhere is limit defined. The fundamental relations of increments, derivatives and differentials are well presented. We are glad to find theorems on infinitesimals which are needed in setting up problems and which are frequently neglected when the calculus is approached from the standpoint of limits.

The part on differential equations seems too short. Integration by partial fractions is omitted. An illustrative problem would have emphasized the importance of the constant of integration. Some illustrations of the use of the definite integral are needed to make plausible their statement that "The concept of the definite integral is the most useful concept in the application of the calculus."

The promised parts on "Functions of Two or More Variables," "Numerical Computation," and on "Elementary Dynamics" have not yet appeared so far as the reviewer is aware.

ERNEST B. LYTLE.

Einführung in die höhere Mathematik. By HANS VON MANGOLDT. Vol. I: *Anfangsgründe der Infinitesimalrechnung und der analytischen Geometrie.* 1911. xiv+477 pp. Vol. II: *Differentialrechnung.* 1912. xi+566 pp. Vol. III: *Integralrechnung.* 1914. x+485 pp. Leipzig, S. Hirzel.

THE volumes before us constitute an important contribution toward the solution of a problem which is of great concern to the majority of teachers of mathematics. That problem has to do with the amount and arrangement of the mathematical methods to be included in courses for students of physics and of engineering. We are of course agreed that there must be included the analytical geometry of two and of three dimensions, differential and integral calculus. But though we realize the deficiencies of the traditional college courses in these subjects, we are not well agreed as to the remedy. There are on the one hand those who claim that the student in question need only learn when and how certain formulas should be used, on the other hand those who would teach all mathematical courses merely as if they were an

end in themselves, with no regard at all to the future interests of the student. Between these extremes the advocates of the middle course have had a hard time. Many more or less successful attempts have been made to solve the problem by combinations of the old courses, eliminating some of the established chapters here and there, and introducing the fundamental notions of analytic geometry and the calculus at earlier stages, mainly with a view to saving time that had hitherto been employed either in solving problems in several different ways, or in the study of subjects which had no immediate application to physics and engineering. Most of the combinations which have come to the reviewer's attention seem to him to fall short in the lack of a central idea which should weld the whole course into an harmonious whole. The old courses have been mixed without being unified.

The book under review is centered on the differential and integral calculus. The first volume is introductory. Assuming the student acquainted with elementary geometry, trigonometry and algebra, the author first discusses certain chapters in advanced algebra, viz. permutations and combinations, the binomial theorem, probabilities, determinants, irrational numbers, this latter including a clear and sound exposition of Dedekind's theory. These matters occupy 177 pages. He then takes up the fundamental notions of analytic geometry, with emphasis on the idea of a function, concluding the volume with a full account of limits and continuity. The chapters on analytic geometry include both two and three dimensions, but are confined to problems on lines and planes, leaving all curved lines and surfaces to be treated in the succeeding volumes in connection with the calculus. The chapter on functions is unusually full and thoroughly modern, based on a brief, but clear, exposition of point-sets. In this chapter are included the interpolation formulas of Lagrange and Newton.

The second and third volumes are characterized by the same thoroughness of treatment and soundness of method that I have tried to indicate in this too short account of the first volume. In the second volume, the subject of infinite series receives a full algebraic treatment in a chapter of 75 pages in addition to a long chapter on Taylor's theorem. In the case of functions of several variables, the condition for validity of $\delta^2 f/\delta x \delta y = \delta^2 f/\delta y \delta x$ is adequately presented. The

volume closes with 87 pages on functions of a complex variable, including the fundamental theorem of algebra and conformal representation.

The third volume, on the integral calculus, includes such matters as Cauchy's theorem, the theorem of Gauss on transformation of a volume integral into a surface integral, Green's and Stokes's theorems, with a short discussion of vectors in connection with the latter. There is an admirable chapter on improper integrals, that is, on integrals of discontinuous functions and integrals with infinite limits. The final chapter, on differential equations, is confined to those equations which commonly occur in the applications, but contains a good treatment of the geometric interpretation of a differential equation and the existence proof.

The volumes contain very much more material than could possibly be included in any ordinary university course even in Germany, but at the same time there is scarcely anything which should not be of essential value to every student of physics or of engineering. As a supplementary hand-book, to which the teacher could refer for a sound and clear discussion of fundamental principles, it is all that could be desired, and is quite the best book of this kind, so far at least as the students referred to are concerned, which has come to the reviewer's attention.

M. W. HASKELL.

Functions of a Complex Variable. By E. J. TOWNSEND. New York, Henry Holt and Company, 1915. vii + 384 pp. 8vo. Price $4.00.

THERE has been a noticeable dearth of text-books in the English language on the elements of the theory of functions of a complex variable. The student with an easy command of German and French has found a rich and delightful literature, while his companion still in the period of language difficulties has had little opportunity to choose his reading in function theory in accordance with his individual tastes and requirements. But within the last three years there have appeared English translations of the classic works of Burkhardt and Goursat, a very full volume by Pierpoint, and the book we have before us for review.

The most obvious advantage of Townsend's treatment seems to be the absence of the synoptic character common to

texts written for advanced students of mathematics. Apparently no knowledge of matters beyond the elementary calculus is assumed, unless the existence of the definite integral $\int_a^b f(x)dx$, where $f(x)$ is a continuous function of the real variable x in the interval $a \leqq x \leqq b$, may be regarded as such an assumption. The general theory of the line integral based on the foregoing existence theorem is completely developed. In this connection it may be remarked that the author's use throughout the book of an *ordinary* curve (precedent is cited for the terminology, page 47) as the path of integration makes it practicable to attain complete rigor in the proofs with a minimum of complexity. The ordinary curve has as its most convenient property the quality of being monotone by segments finite in number, a property not possessed by the more general *regular* curve used in Osgood's treatise. To the beginner the fundamental theorems of function theory are fully as satisfactory stated in terms of the simpler curve. The zest for generalization naturally comes at a later period in his study. We have here then a treatment of the function theory which appears to go to the bottom of things without being led into delicate generalizations and which requires on the part of the reader only a thorough first course in calculus and a certain maturity of mind.

The book is unique in its extraordinarily detailed treatment of the elementary functions (55 pages). It thus affords opportunity for an intimate acquaintance with old friends from a new point of view and offers an interesting concrete field of application for many of the general theorems. This is all the more important in view of the fact that "the material chosen deals for the most part with the general properties of functions of a complex variable, and but little is said concerning the properties of some of the more special classes of functions, as for example elliptic functions, etc." It may well be questioned whether a first course in function theory should include among its objects the introduction of a variety of new functions or whether the emphasis should be rather on a more profound acquaintance with the old functions and a clearing up of difficulties naturally arising in the study of the calculus, incidentally laying foundations for work in the field of elliptic functions, etc., if the occasion for such work presents

itself. The latter seems to be distinctly the view-point of the author of the present volume.

Chapter I is devoted to the arithmetic of complex numbers including the geometric representation of the processes of addition, subtraction, multiplication, and division. Chapter II is concerned with the notion of a limit in the complex domain. A number of theorems useful later on are established. For example, it is proved that if $f(z)$ is continuous in a closed region, this continuity is uniform—a theorem not always presented in books of an elementary character. It is in this chapter only that one finds the author yielding to the temptation to generalize beyond the immediate needs of the student. The definition given of continuity in a boundary point of a region is such that the continuity of a function in a closed region does not imply by definition the continuity of the set of boundary values itself. Instead of merely supplying the proof that the set of boundary values is actually continuous, the author states and proves the more general theorem (page 38) that "if $f(z)$ is defined for a closed region S and converges uniformly along an arc C of the boundary of S, then $f(t)$ is continuous, where t denotes the values of z on C." So far as later applications are concerned this theorem seems to be needless. However, the whole chapter is a vigorous exercise in ϵ, δ reasoning and the added refinement may increase its disciplinary value.

Chapter III, a long chapter devoted to the fundamental integral theorems, is one of the most interesting in the book. The existence of a definite integral of a continuous function $f(z)$ over an ordinary curve C having been once established by means of the relation $\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$, $f(z)$ is thereafter kept intact. The Cauchy-Riemann differential equations are not given until near the end of the chapter and are then derived by means of the very integral theorems which usually depend on them. The development of the integral theorems starts with the Goursat proof of the Cauchy theorem, or as the author calls it, the Cauchy-Goursat theorem, a title which seems proper since Goursat really contributed a new theorem rather than a new proof of an old theorem. Then follows the derivation of the Cauchy integral formula and the proof that the mere existence of a derivative of a

function of a complex variable in a region implies the continuity of the derivative.

Morera's converse of the Cauchy theorem is given particular prominence and finds wide application. Using this theorem it is easily shown that the necessary and sufficient condition for a continuous function $f(z) = u + iv$ to be holomorphic is either that $f(z)$ satisfy the integral equation $\int_C f(z)dz = 0$ or that u and v have continuous first partial derivatives satisfying the Cauchy-Riemann differential equations. The integral theorem renders unnecessary the usual ϵ, δ proof of the sufficiency of the differential equations. The chapter closes with the usual theorems concerning Laplace's differential equation and a brief discussion of its significance in mathematical physics.

In Chapter IV the elementary functions are mapped in greater detail than in any other English text with which the reviewer is familiar. The definition of the exponential function as

$$e^z = \lim_{n=\infty}\left(1 + \frac{z}{n}\right)^n$$

and the definitions of the sine and cosine functions as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

permit the discussion of the properties of these functions in advance of the introduction of power series. In addition to the usual elementary functions several particular cases of the function

$$w = \log\frac{(z - \alpha_1)^{k_1}(z - \alpha_2)^{k_2}\cdots(z - \alpha_n)^{k_n}}{(z - \beta_1)^{\lambda_1}(z - \beta_2)^{\lambda_2}\cdots(z - \beta_n)^{\lambda_n}}$$

having important physical applications are mapped and the physical interpretations pointed out.

Chapter V, on linear fractional transformations, is sufficiently full to serve as a foundation for the theory of automorphic functions. The figures in illustration of the process of stereographic projection are unusually clear and accurate and should be a great aid at this point to the student's imagination.

Chapter VI, on infinite series, can be criticized only on the ground that it comes so late (approximately at the beginning

of the second half of the volume). The author's purpose seems to have been to do first everything that can be conveniently done without the use of power series and then, having once introduced power series, to make rapid and continual use of them in developing the general analytic function theory. Logically this method is beyond criticism and it may have pedagogical advantages. In this chapter the usual theorems on power series are developed, closing with Taylor's expansion of a function holomorphic in a given region.

Chapter VII takes up the theorem on Taylor's expansion and out of it develops the notion of analytic continuation and the general definition of an analytic function. This discussion is as clear cut and accurate as one often finds. Particularly to be noted, as serving to dispel any hazy notions the student may have at this stage, is the comparison by means of examples of the properties of functions of a real variable which are differentiable infinitely many times with the properties of analytic functions of a complex variable—also differentiable infinitely many times. For example, the function

$$f(x) = e^{-1/x^2},$$

considered zero when x is zero, has infinitely many derivatives in the real domain for all real values of x. Nevertheless

$$\lim_{x=0} \frac{f(x)}{x^k} = 0,$$

where k is any integer however large. This it is pointed out is in sharp contrast to the behavior of a single valued analytic function at a zero point, in that the zero points of the latter are necessarily of definite positive integral order. Several examples, such as the series

$$E(z) = \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \cdots,$$

convergent except for values of z upon the unit circle, are given to make clear the distinction between an analytic function and a function defined by an analytic expression.

The notion of analytic function having been established, zero points, poles, and essential singular points are defined and the usual theorems are included. Regular and singular points at infinity, Laurent's expansion, and the theory of

residues are next in order. We then have a full discussion of the properties of rational functions, with a proof of the fundamental theorem of algebra. Passing to transcendental functions, the Mittag-Leffler theorem in its simplest form is proved. An innovation in connection with this proof consists in the introduction of a figure as an aid to the reader in following the analytic reasoning. The function $\zeta(z)$ and the elliptic function $\wp(z) = -\zeta'(z)$ are used in illustration of the theorem. Such properties of infinite products as are essential to an understanding of Weierstrass's primary factors are developed. The chapter closes with a brief discussion of the properties of simply and doubly periodic functions.

Chapter VIII, the last, is devoted to a brief but fairly comprehensive treatment of the properties of multiple-valued functions.

On the whole the book is well coordinated with our undergraduate courses and covers just about the ground in function theory which the first year graduate student of mathematics should get well in hand.

H. B. PHILLIPS.

Plane and Spherical Trigonometry and Tables. By G. WENTWORTH and D. E. SMITH. Boston, Ginn and Company, 1914. 230 + 104 pp.

QUOTING from the preface, this is "a work to replace the Wentworth Trigonometry which has dominated the teaching of the subject in America for a whole generation." . . . "With respect to sequence the rule has been followed that the practical use of every new feature should be clearly set forth before the abstract theory is developed."

In several particulars the book could be made more useful for students intending to pursue mathematics further. For instance, no mention has been made of Argand's diagram or of hyperbolic functions, though the logarithms of negative numbers are unusually well treated. This excellence is balanced by the unfortunate use of negative characteristics which will lead the student into endless trouble later.

Inverse function theory merits more extensive treatment even at the cost of fewer examples, but the related general formulas for all angles having the same sine, cosine or tangent are to be commended. The small pink representations of coordinate paper are attractive to the eye, but the graphs of

the trigonometric functions are too small and not accurate. One full page graph with carefully distinguished curves would be more satisfactory. In the early pages the definition of coordinates is correct, but unfortunately all notion of directed lines is abandoned in the proofs of the summation formulas for sines and cosines.

Reduction of functions to the first quadrant, which in many books is needlessly diffuse, is here given in a compact form which lacks only a good rule as a summary to be quite sufficient. The double inequality in the proof for the sines of small angles follows the usual course, and as in most of our text-books does not show that the two tangents are longer than the arc. The corresponding S and T tables have been made relatively inconspicuous and supplemented by more useful tables of functions of small angles given for each second.

In the discussion of variation of trigonometric functions double signs have been used to advantage with the zero values as well as with the infinity sign. The solid perspective figures in the latter part of the book are good, and at the end is found, what is none too common in our text-books, an index.

As a whole the book does not differ noticeably from the many text-books covering the same ground. Examples are particularly abundant while the theoretical portions might well be amplified and improved.

F. H. SAFFORD.

Wilhelm Schells Allgemeine Theorie der Kurven Doppelter Krümmung. Dritte Auflage neu bearbeitet von ERICH SALKOWSKI. Leipzig, B. G. Teubner, 1914. 196 pp. Price 8 marks.

THE first edition of this book appeared in 1859, the second in 1898. It was the purpose of the author to develop the elements of the theory of curves of double curvature by the geometric method of infinitesimals without recourse to higher analysis. Since the investigation included the study of the three surfaces generated by the tangents, principal normals, and binormals of a general curve, there appears incidentally a considerable amount of material concerning the geometry of a surface in the neighborhood of an ordinary point. Consequently a reader (who might easily be an undergraduate) could obtain from the book a very good idea of the elements of the differential geometry of curves and surfaces.

Any one familiar with Salkowski's papers in this field would recognize his fitness to prepare a third edition of the book. For a number of his papers are written from its point of view. Many sections have been rewritten and there are a number of essential additions. A notable one is the proof of the fundamental theorem concerning the determination of a curve by its intrinsic equations. A new chapter is devoted to a full discussion of skew ruled surfaces, in preparation for the study of those generated by the principal normals and binormals of a curve.

When two curves in space are in one-to-one correspondence such that corresponding tangents, principal normals, and binormals respectively are parallel, the curves are in the relation of a transformation of Combescure—to use the terminology due to Bianchi. One readily thinks of other correspondences involving parallel arrangements, as well as ones having to do with differential quantities. An investigation of these questions appears in the new edition for the first time, as do also the ideas and properties of successive evolutes.

A student reading one of the advanced treatises on differential geometry will do well to consult this little book, because of its method of attack and its richness in material.

LUTHER PFAHLER EISENHART.

Ten British Mathematicians. By ALEXANDER MACFARLANE. No. 17, Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. New York, John Wiley and Sons, 1916. 148 pp. Price $1.25.

THIS is somewhat of a departure from the preceding sixteen numbers of this series in subject matter, being an account of individual mathematicians and their contributions to mathematics, rather than a survey of some particular field of the subject. The ten men are George Peacock, Augustus De Morgan, Sir William Rowan Hamilton, George Boole, Arthur Cayley, William Kingdon Clifford, Henry John Stephen Smith, James Joseph Sylvester, Thomas Penyngton Kirkman, and Isaac Todhunter. The author had the advantage of personal acquaintance with a number of the men of whom he wrote, as well as the interest of having been born a fellow countryman. He has given us the life history of the man without too much detail, and yet with enough intimacy, so that we have a picture of the man as a man, as well as a scientist.

The book consists of ten lectures delivered to audiences composed of students, instructors, and townspeople at Lehigh University during the years 1901 to 1904. As each was delivered as a separate lecture, there is no direct connection between them. The critical analysis of the contributions of each mathematician is probably as well done as would be possible considering the character of the audience and the length of time at the disposal of the speaker. We get a very good general notion of what each contributed and its value to the science.

The ten men chosen are not equally well known to American mathematicians. Certainly we have a deeper interest in and knowledge of Sylvester than of the others. Todhunter would be best known for his many textbooks, and De Morgan for his Paradoxes. It is doubtful if American mathematicians know the English as well as the continental mathematicians, especially the German. American students of mathematics who have gone abroad for study have gone chiefly to the German universities. This has tended to bring American mathematicians into closer relations with the German mathematicians and their work than with the English. For that reason, this book on English mathematicians is valuable.

There is a tendency today to try to give some comprehension of the problems and methods and achievements of mathematics to those who are not specialists in mathematics. A recent book by Professor G. A. Miller, of the University of Illinois, is such an effort. These lectures by Dr. Macfarlane serve that purpose, as they were not prepared for specialists in mathematics alone.

We have a partial promise in the preface that there will be a later volume dealing with ten mathematicians who worked chiefly in applied mathematics. We hope that this promise may be speedily fulfilled.

The reading of the page proofs does not seem to have been done carefully enough. There are too many letters that have dropped out. Otherwise this volume has the same convenient size and neatness of appearance that characterize the other volumes of this set.

THOS. E. MASON.

Das Perpetuum Mobile. By F. ICHAK. Bändchen 462 of the series: Aus Natur und Geisteswelt. Leipzig, Teubner, 1914. 104 pp. Price M. 1.25

THE question of perpetual motion has interested mankind for many centuries. The author says: "We see before us an almost endless series of attempts of different sorts which have one thing in common—a negative result. But mankind does not let itself be convinced of the impossibility of the undertaking by these unnumbered failures: here the wish is mightier than the judgment."

This little book does not claim to be an exhaustive treatise on the subject, but does cover the field remarkably well in such short space. This is largely due to the systematic arrangement of the material. Perpetual motion is discussed under three heads: first, that perpetual motion which depends upon the creation of energy by some device, and this is what is usually meant by perpetual motion; second, that which does not create energy but transforms it from one form to another in some cycle; and third, that motion which is not perpetual but of indefinite length, like the motion of the radium clock. The first of these is the oldest and has been the goal of the greatest number of perpetual motion seekers. The efforts to make a device for the creation of energy which could be utilized in doing work are discussed under several types, such as mechanical, magnetic, hydraulic perpetual motion, etc. Type machines and machines which have attracted unusual attention are described in detail and often illustrated. The effect of learning and the development of science upon the efforts at finding such a machine are discussed as well as the reaction of these experiments on science. With the theory of the conservation of energy came the attempts to secure perpetual motion of the second sort mentioned. The perpetual motion machines of the third sort, or apparent perpetual motion machines, like the radium clock or the radiometer, do not create energy but have it furnished from nature, and while the supply is very great it is not inexhaustible. The author closes with a section on perpetual motion in the future, in which among other things is mentioned the possibility of getting around the difficulties of the conservation of energy by a new theory concerning energy. There is also a prophecy, likely to be fulfilled, that we shall always have the perpetual motion seekers with us.

THOS. E. MASON.

Constructive Geometry. Exercises in elementary geometric drawing. Prepared under the direction of EARLE RAYMOND HEDRICK. New York, Macmillan Company, 1916. 4to. 72 pages; 43 figures.

THIS book is a manual in very elementary work in geometric drawing. The scheme is first a discussion of a problem, then a number of exercises, finally a number of blank pages on which to draw the exercises. It begins with divisions of a straight line, and emphasizes neatness and accuracy in each detail.

Exercises on the circle include a considerable number in design, all of which are very elementary. While this drill is in the right direction, it is almost too simple to be of much value. In fact one of the greatest needs of our instruction in geometry is a more comprehensive treatment of construction by means of ruler and compass, including a full discussion of the graphical solution of quadratic equations.

VIRGIL SNYDER.

Grundzüge der Perspektive nebst Anwendungen. By Professor KARL DOEHLEMANN, of the technical school at Munich. (Aus Natur und Geisteswelt, volume 510.) Leipzig, Teubner, 1916. 104 pages, 91 figures, and 11 photographs.

THE volumes of this series are to be written in a popular style, suitable for readers with little or no previous training in the specific line discussed by each volume. Thus, in the present case no use is made of solid geometry, and only the most elementary properties of similar polygons in plane geometry are assumed as known. Probably for this reason the introductory chapter strikes a reader as heavy and prolix, although the method of introduction is unusually well chosen. The author finds it necessary to present the essentials of the $h - v$ orthogonal projection, but only in so far as points and straight lines are concerned, and these are employed immediately to produce a perspective picture.

The concept of the vanishing point and vanishing line is developed in great detail. No exercises are proposed for the reader, except that an occasional construction is left to be completed, but on the other hand a large number of drawings and several reproductions of paintings are given, to assist in illustrating the point under discussion. Apart from the very brief explanation of the two-plane orthogonal projection, the

discussion of figures bounded by planes is treated in sufficient detail. A short discussion of the perspective of the circle is added, and a still shorter discussion of shades and shadows. The book has a table of contents, an index, a list of books for further reading, and is almost entirely free from typographical errors. It is well fitted for its purpose of providing a knowledge of the essentials of perspective in a small popular volume.

VIRGIL SNYDER.

A Treatise on Statics. By GEORGE M. MINCHIN. Vol. II. Fifth Edition. Revised by H. T. GERRANS. Oxford, The Clarendon Press, 1915.

"IN December, 1913, when the preparations for a new edition became necessary, it was suggested to the author (1) that certain portions should be omitted (in the hope that they might form the basis of a separate work); (2) that some account of the author's recent researches in spherical harmonics should be given; (3) that a substantial number of examples should be added." The author determined which chapters and articles should be left out, but his death occurred before the task of revision was begun.

In this latest edition 467 examples have been added in an appendix, also a very few in the text, while the original numbering of the articles in the fourth edition has been retained. Three entire chapters (of the fourth edition) have been omitted, Chapter XVIII, Analysis of Strains and Stresses, Chapter XIX, Electrostatics, also the chapter on Astatic Equilibrium, viz., Chapter XIV, in which the subject was treated with the aid of quaternions.

Other omissions are arts. 221–226, 228–234, on reciprocal screws, etc., arts. 280, 281, on general properties of static energy and stable configuration, arts. 288–296 on elastic wires, inextensible surfaces and liquid surfaces, arts. 307–314 on tortuous curves and kinetic analogues. Note A, on the equation of capillarity, has disappeared, as well as the index to Volume II.

Otherwise this edition is a verbatim reproduction of the fourth edition, with occasional obvious omissions of subscripts. In art. 235 the correct reading is $\phi = qAR$ and $K = pR$. In the last equation in art. 241, the $-$ sign should be $+$, and in equation (2) of problem 9 at the end of the same article α/μ should be a/μ.

Certain errata found in the fourth edition, all but one of which were noted by Professor Hoover of Ohio University, have been corrected, except that a superfluous 1 still remains in the answer to problem 12, art. 325. Other errors uncorrected in both editions are $W/2$ in place of W as the answer to problem 5, art. 205; and in art. 227 the variables should be connected by *three* equations if the body has three degrees of freedom.

It is evident that this edition is essentially a reprint, and that the reviser felt a commendable reluctance in changing in any way, except as sanctioned by the author, a text which holds such a high reputation among scholars.

F. H. SAFFORD.

NOTES.

AT the meeting of the Edinburgh mathematical society on November 10 the following papers were read: By E. T. WHITTAKER: "Some theorems on determinants"; by H. DATTA: "On the theory of continued fractions."

THE National academy of sciences held its autumn meeting at the Massachusetts Institute of Technology on November 13–15. The programme included the following mathematical papers: By F. R. MOULTON: "On analytic functions of infinitely many variables"; by H. S. WHITE, F. N. COLE, and LOUISE D. CUMMINGS: "Enumeration of all triad systems on fifteen elements"; by W. E. STORY: "Some variable three-term scales of relation." The list of scientific exhibits included graphic representations of triad systems, by H. S. WHITE.

THE Association of mathematics teachers of New Jersey held its fifth regular meeting at Newark on November 25th. The programme included the final report of the committee on trigonometry courses and papers by: A. W. BELCHER: "The first year high school course in mathematics"; W. D. REES: "Newton's analytical triangle"; C. R. MACINNES: "Some theorems on regular polygons described on the sides of a triangle"; WILLIAM STRADER: "Teaching first year algebra."

THE two Benjamin Peirce instructorships in mathematics at Harvard University (see BULLETIN, volume 21, page 315) are again open to general competition. Applications for the year 1917–1918, accompanied by the necessary papers, should be sent to Professor Bôcher, who will be glad to furnish further particulars. All applications must be filed by February 1, 1917.

A complete list of American doctorates conferred during the academic year 1915–1916 has been compiled for *School and Society*. From advanced proof sheets kindly furnished by Professor Cattell it appears that the total number during this period was 603, of which 327 may be assigned to the sciences. The list of mathematical doctorates, 34 in number, follows below, the title of the dissertation being given in each case.

J. W. ALEXANDER, II, Princeton, "Functions which map the interior of the unit circle upon simple regions"; P. M. BATCHELDER, Harvard, "The hypergeometric difference equation"; R. W. BRINK, Harvard, "Some integral tests for the convergence and divergence of infinite series"; A. F. CARPENTER, Chicago, "Ruled surfaces whose flecnode curves have plane branches"; L. C. COX, Cornell, "The finite groups of birational transformations of a net of cubics"; J. V. DEPORTE, Cornell, "Irrational involutions on algebraic curves"; W. VAN N. GARRETSON, Michigan, "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution"; R. E. GILMAN, Princeton, "On the summability by the Cesaro method of partial derivatives of Laplace series"; A. M. HARDING, Chicago, "On certain loci projectively connected with a given plane curve"; W. L. HART, Chicago, "Differential equations and implicit functions in infinitely many variables"; M. G. HASEMAN, Bryn Mawr, "Knots with a census of the amphicheiral with twelve crossings"; J. O. HASSLER, Chicago, "Plane nets periodic of period three under the Laplace transformation"; ARCHIBALD HENDERSON, Chicago, "The twenty-seven lines upon the cubic surface"; J. R. KLINE, Pennsylvania, "Double elliptic geometry in terms of point and order"; PEYSAH LEYZERAH, Clark, "On the indeterminate linear inequality with irrational coefficients"; G. H. LIGHT, Yale, "The dependence of the topography of envelopes of systems of ex-

tremals on curvature"; P. H. LINEHAN, Columbia, "Contributions to equilong geometry"; F. J. MCMACKIN, Columbia, "Some theorems in the theory of summable divergent series"; A. L. MILLER, Harvard, "Systems of pencils of lines in ordinary space"; NORMAN MILLER, Harvard, "Some problems connected with the linear connectivity of manifolds"; F. D. MURNAGHAN, Johns Hopkins, "The lines of electric force due to a moving electron"; J. R. MUSSELMAN, Johns Hopkins, "A set of eight self-associated points in space"; A. L. NELSON, Chicago, "Plane nets with equal Laplace-Darboux invariants"; T. A. PIERCE, California, "The numerical factors of the arithmetic forms"; C. H. RAWLINS, JR., Johns Hopkins, "Complete systems of concomitants of the three-point and the four-point in elementary geometry"; S. W. REAVES, Chicago, "Metric properties of flecnodes on ruled surfaces"; A. R. SCHWEITZER, Chicago, "Les idées directrices de la logique génétique des mathématiques"; T. M. SIMPSON, Wisconsin, "On a functional equation of Abel"; E. S. SMITH, Virginia, "Solution of the equation of secular variation by a method due to Hermite"; PAULINE SPERRY, Chicago, "Properties of a certain projectively defined two-parameter family of curves on a general surface"; J. H. WEAVER, Pennsylvania, "Some extensions of the work of Pappus and Steiner on tangent circles"; MARY E. WELLS, Chicago, "The determination of all inequalities of certain types in general linear integral equation theory"; A. R. WILLIAMS, California, "A birational transformation connected with a pencil of cubics"; C. H. YEATON, Chicago, "Surfaces characterized by certain properties of their directrix congruences."

THE Royal Society has awarded the Sylvester medal to Professor GASTON DARBOUX, of the University of Paris.

DR. A. SIGNORINI, of the University of Padua, has been appointed associate professor of rational mechanics at the University of Palermo.

AT the U. S. Naval Academy DR. R. E. ROOT has been promoted to a full professorship of mathematics and mechanics.

PROFESSOR J. F. REILLY, of the State University of Iowa, has been promoted to an associate professorship of mathematics.

At the University of Washington Dr. E. T. Bell has been promoted to an assistant professorship of mathematics.

At Indiana University Dr. Cora B. Hennel has been promoted to an assistant professorship of mathematics.

Mr. C. W. Wester, of the State University of Iowa, has been appointed assistant professor of mathematics in the Iowa State Teachers College.

Dr. D. F. Barrow, of the University of Georgia, has been appointed instructor in mathematics in the Sheffield Scientific School.

Professor R. M. Barton, of Lombard College, has been appointed instructor in mathematics in the University of Minnesota.

Dr. A. L. Miller has been appointed instructor in mathematics in the University of Michigan.

Professor G. Pennacchietti, of the University of Catania, died August 21 at the age of sixty-six years.

Assistant professor J. C. Rayworth, of Washington University, died November 11 after an illness of several months.

Dr. Henry Gunder, formerly professor of mathematics at Findlay College, died November 25 at the age of seventy-nine years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

HOPKINS (L. A.). See ZIWET (A.).

LECAT (M.). Table des matières des tomes I à XX (1894–1913) de *L'Intermédiaire des Mathématiciens*. Paris, Gauthier-Villars, 1916. 8vo. 4+280 pp. Fr. 8.00. To subscribers to *L'Intermédiaire* for 1915 and 1916, Fr. 4.00.

LOREY (W.). Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19. Jahrhunderts. (Abhandlungen über den mathematischen Unterricht in Deutschland veranlasst durch die internationale mathematische Unterrichtskommission, Band III, Heft 9.) Leipzig, Teubner, 1916. Gr. 8vo. 16+440 pp. Geh. M. 12.00

LOVE (C. E.). Differential and integral calculus. New York, Macmillan, 1916. 8vo. 18+343 pp. Cloth. $2.10

RIGGS (N. C.). Analytic geometry. 9th reprint. New York, Macmillan, 1915. 12mo. 294 pp. $1.60

ZIWET (A.) and HOPKINS (L. A.). Analytic geometry and principles of algebra. 3d reprint. New York, Macmillan, 1915. 12mo. 369 pp. $1.60

II. ELEMENTARY MATHEMATICS.

ANDOYER (H.). Nouvelles tables trigonométriques fondamentales contenant les valeurs naturelles des lignes trigonométriques de centième en centième du quadrant avec vingt décimales, de neuf en neuf minutes avec dix-sept décimales, et de dix en dix secondes avec quinze décimales. Ouvrage publié à l'aide d'une subvention accordée par l'Université de Paris (Fondation Commercy). 2 tomes. Paris, Hermann, 1915–1916. 4to. Tome 1. 18+342 pp. Tome 2. 3+275 pp. Fr. 24.00+18.00

BALTIN (R.). See MÜLLER (H.).

BARDEY (E.). See LIETZMANN (W.).

BRUFF, HAYDEN and WATKINS. Practical primary arithmetic. Philadelphia, Lippincott, 1916.

ELSON (W. H.). See SHORT (R. L.).

HAYDEN. See BRUFF.

LIETZMANN (W.) und BARDEY (E.). Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe für höhere Knabenschulen von W. Lietzmann und P. Zühlke. A: für Gymnasien. 1ter Teil: Unterstufe, von W. Lietzmann. 2te Auflage. Leipzig, Teubner, 1915. Gr. 8 vo. 6+202 pp. Geb. M. 2.00

MAHLERT (A.). See MÜLLER (H.).

MAIWALD (W.). See MÜLLER (H.).

MULLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe A: fur Gymnasien und Progymnasien. 1ter Teil: Die Unterstufe. 6te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 8+144 pp. Geb. M. 2.20

——. Die Mathematik auf den Gymnasien und Realschulen. Ausgabe B: für reale Anstalten und Reformschulen. 2ter Teil: Die Oberstufe. 1te Abteilung: Planimetrie, Algebra, Trigonometrie und Stereometrie. 5te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 11+320 pp. Geb. M. 3.00

MULLER (H.), BALTIN (R.) und MAIWALD (W.). Kurzgefasstes Lehrbuch der Mathematik fur Seminare und Praparandenanstalten. 5te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 10+256 pp. Geb. M. 2.60

MÜLLER (H.), BALTIN (R.) und SEGGER (F.). Rechenbuch fur Praparandenanstalten. 1ter Teil· Lehraufgabe der 3te Klasse. 4te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 6+184 pp. Geb. M. 1.80

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch fur Lyzeen (hohere Madchenschulen). 1ter Teil: Arithmetik und Algebra. 5te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 4+160 pp. Geb. M. 2.00

MULLER (H.) und PIETZKER (F.). Rechenbuch fur die unteren Klassen der hoheren Lehranstalten. Ausgabe in 3 Heften. Heft 3: fur Quarta. 4te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 6+122 pp. M. 1.00

MULLER (H.) und SEGGER (F.). Rechenbuch fur die Vorschule. Heft 1: Lehraufgabe des ersten Schuljahres. 5te Auflage. Leipzig, Teubner, 1915. 72 pp. Gr. 8vo. M. 0.80

PHILIP (M.). Algebra. New York, Hinds, Hayden and Eldredge, 1916. 8vo 10+201 pp.

PIETZKER (F.). See MULLER (H.).

SAMESON (C. H.). Algebra review. New York, World Book Company, 1916. · 8vo.

SEGGER (F.). See MÜLLER (H.).

SHORT (R. L.) and ELSON (W. H.). Introduction to mathematics. (Junior High School Series.) Boston, Heath, 1916. 8vo. 8+200 pp.

WATKINS. See BRUFF.

ZUHLKE (P.). See LIETZMANN (W.).

III. APPLIED MATHEMATICS.

ATHENAEUM subject index to periodicals, 1915. Science and technology, with special reference to the war in its technological aspects. London, The Athenaeum, 1916. 80 pp. 2s. 6d.

BREED (C. B.) and HOSMER (G. L.). Principles and practice of surveying. Vol. 1: Elementary surveying. 4th edition. Vol. 2· Higher surveying. 2d edition. New York, Wiley, 1915. 20+591+18+443 pp. Leather. $3.00+2.50

BURGATTI (P.). Lezioni di meccanica razionale. Bologna, Zanichelli, 1916. 8vo. 5+501 pp. L. 18.00

CARPENTER (C. K.). See WOLFARD (M. R.).

CLASSEN (J.). See GRIMSEHL (E.).

DRZEWIECKI (S.). See JOUKOWSKI (N.).

GEITEL (H.). See GRIMSEHL (E.).

GIANFRANCESCHI (G.). La fisica dei corpuscoli: molecole, atomi, elettroni. Torino, Fratelli Bocca e Roma, tip. Artigianelli, 1916. 8vo. 7+239 pp. L. 5.00

GRIMSEHL (E.). Lehrbuch der Physik. 2ter Band. 3te Auflage, durchgesehen und erganzt von J. Classen, H. Geitel, W. Hillers und W. Koch. Leipzig, Teubner, 1916. Gr. 8vo. 10+542 pp. Geh. M. 7.00

HILLERS (W.). See GRIMSEHL (E.).

HOSMER (G. L.). See BREED (C. B.).

HOWE (C. B.). Mechanical drafting. New York, Wiley, 1915. 10+147 pp. Cloth. $1.75

JOUKOWSKI (N.). Aérodynamique. Translated by S. Drzewiecki. Paris, Gauthier-Villars, 1916. 18+227 pp. Fr. 11.00

JUDGE (A. W.). The design of aeroplanes. London, Whittaker, 1916. 9s.

KAPP (G.). Principles of electrical engineering and their applications. Vol. 1: Principles of electrical engineering. New York, Longmans, 1916. 8vo. 12+356 pp. $4 25

KENT (R. T.). Power transmission by leather belting. New York, Wiley, 1916. 6+114 pp. $1.25

KOCH (W.). See GRIMSEHL (E.).

PERKINS (L.). The railroad taper. The theory and application of a compound transmission curve based upon thirty-foot chords. New York, Wiley, 1915. 10+355 pp. Leather. $2.50

POPOVATZ (P.). Notions générales sur les appareils à réaction. Paris, Gauthier-Villars, 1916. 8vo. 36 pp. Fr. 1.50

VATER (R.). Maschinenelemente. (Aus Natur und Geisteswelt.) 2te Auflage. Leipzig, Teubner, 1915. Geb. M. 1.25

VITI (R.). Elementi di scienza attuariale per gli istituti tecnici. Teorie matematiche elementari della finanza e della previdenza. Bologna, Rocca S. Casciano, 1916. Gr. 8vo. 10+285 pp. L. 7.00

WERKMEISTER (P.). Feldmessen und Nivellieren. 2te verbesserte Auflage. Berlin, 1915. M. 0.90

WOLFARD (M. R.) and CARPENTER (C. K.). Conversion chart. New York, Wiley, 1915. 12 by 34 inches. $0.25

LIBRARY.

THE TWENTY-EIGHTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-eighth regular meeting of the San Francisco Section of the Society was held at the University of California on Saturday, November 25, 1916. The total attendance was nineteen, including the following members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor L. M. Hoskins, Dr. Frank Irwin, Professor C. J. Keyser, Professor D. N. Lehmer, Professor W. A. Manning, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam.

Professor D. N. Lehmer was elected chairman of the Section and presided at the meeting. Professor W. A. Manning was elected secretary and Professors Blichfeldt, McDonald, and Manning members of the programme committee of the Section for the ensuing year.

The following papers were presented at this meeting:

(1) Professor W. A. MANNING: "The degree and class of multiply transitive groups."

(2) Professor W. A. MANNING: "The primitive groups of class fifteen."

(3) Miss EDNA JOHNSTON: "Graphical determination of the nature of the roots of cubic and quartic equations."

(4) Miss MAURINE LABER: "Concerning the combinations of 36 elements 6 at a time."

(5) Professor A. F. CARPENTER: "Some fundamental relations in the projective differential geometry of ruled surfaces."

(6) Professor C. J. KEYSER: "A note concerning groups of dyadic relations."

(7) Professor L. M. HOSKINS: "The theory of strain of an elastic solid from a condition of great initial stress; with application to two cases of strain of a gravitating sphere."

(8) Mr. J. S. TAYLOR: "Complete existential theory of Bernstein's set of four postulates for Boolean algebras."

(9) Professor H. F. BLICHFELDT: "Linear groups which contain transformations having only two distinct characteristic roots."

The papers of Miss Johnston and Miss Laber were communi-

cated to the Society by Professor Griffin. Mr. Taylor was introduced by Dr. Bernstein. In the absence of the authors the papers of Miss Johnston, Miss Laber, and Professor Carpenter were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Manning presented the following theorem: Let n be the degree and μ the class of a multiply transitive substitution group G that contains a substitution of order 2 and degree μ; then if the transitivity of G is

double, $\mu > \frac{1}{2}n - \frac{1}{2}\sqrt{n} - 1$;
triple, $\mu \geqq \frac{1}{2}n$;
4-ple, $\mu \geqq \frac{1}{2}(n+1)$;
5-ple, $\mu \geqq \frac{3}{5}n$;
6-ple, $\mu \geqq \frac{3}{4}n$;
7-ple, $\mu \geqq \frac{10}{13}n$;
more than $(p_1 + p_2 + \cdots + p_r)$-ple, where $p_1, p_2, \cdots, p_r$ are distinct odd primes $(r > 1)$,

$$\mu > \frac{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1) - 1}{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)} n.$$

Bochert, in the proof of his theorem, makes the commutator the corner stone of the whole structure. Here the commutator is abandoned and a close study of diedral rotation groups takes its place. The results of that study may be thus summarized: If the order of a group of class μ generated by two substitutions s and t of order 2 and degree μ is divisible by each of the odd primes $p_1, p_2, \cdots, p_r$, its degree n does not exceed $\mu + \mu/(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)$; if the order N of st is odd or is divisible by 4, n does not exceed $\mu + 2\mu/N$; if N (> 2) is twice an odd number, n is not greater than $\mu + 2\mu/N + 8\mu/N^2$.

2. It has been known for some time that the only primitive substitution groups of class $3p$, p an odd prime, that contain a substitution of degree $3p$ and order p are three in number and belong to one class only, class 15. In addition to these groups Professor Manning finds that there are just four other primitive groups of class 15. In fact, if k is any positive whole number greater than three, there are three simply transitive primitive groups of degree k^2 and class $3k$, and one simply

transitive primitive group of degree $\frac{1}{2}(k+1)(k+2)$ and class $3k$. The last is simply isomorphic to the alternating group of degree k. The others are simply isomorphic to the three imprimitive groups of degree $2k$ whose intransitive head is the direct product of two alternating groups of degree k in two different sets of letters, or is the dimidiate of the direct product of two symmetric groups of degree k on distinct letters. An auxiliary theorem suggested by this problem is this: If the degree of a transitive constituent of the subgroup leaving one letter fixed in a simply transitive primitive group exceeds by two (or more) units the degree of any other transitive constituent of that subgroup, then the constituent of highest degree is a simply transitive group.

3. In this note Miss Johnston points out a simple way to determine the nature of the roots of a quartic, in the various possible cases, by considering the points in the complex plane which represent the roots of the reducing cubic and their respective square roots. Similarly for a cubic and its reducing quadratic (sextic). Certain questions relating to discriminants are also discussed.

4. In his Mathematical Essays and Recreations, Ball describes methods of finding $n(n+1)$ combinations of n^2 elements taken n at a time without repetition of any pair, in the cases $n = 4, 5, 7, 8$. In the present note Miss Laber proves this to be impossible in the case $n = 6$, reported by Ball as having "baffled all attempts to find a solution." The proof is based upon a very convenient symmetrical system of notation, and a method of enumeration by which combinations involving common pairs can be rapidly eliminated. The number of admissible combinations resulting from various types of substitutions is noted.

5. In his projective theory of ruled surfaces, Wilczynski makes frequent use of a moving tetrahedron of reference, the homogeneous coordinates of whose vertices, when referred to any fixed system of homogeneous coordinates, are solutions of the system of differential equations

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0.$$

If a point, line or plane is to be fixed in space, its coordinates when referred to this moving tetrahedron must satisfy certain conditions expressible as differential equations, and conversely. Professor Carpenter's paper (*a*) determines necessary and sufficient conditions for fixed points, lines and planes and (*b*) indicates by a number of theorems the usefulness of these conditions in enriching the theory of ruled surfaces. A portion of this paper was communicated to the Chicago Section of the Society at the meeting of April, 1915.

6. The abstract relations studied in symbolic logic fall into various cardinal classes. The relations of a class are combinable in accordance with certain fundamental rules or modes, as logical addition, logical multiplication, relative multiplication. Professor Keyser's paper points out the desirability of examining the more important systems thus arising with a view to ascertaining which of them are groups. The paper presents some results found on the threshold of such an examination. It is found, for example, that under any of the mentioned rules of combination the ensemble of relations that constitute a type (as this term is understood in the Principia Mathematica of Whitehead and Russell) possesses the group property and two but only two of the other properties required in the four-property definition of a group. On the other hand, the ensemble of one-one relations that cover and are restricted to an arbitrarily chosen field of relations is a group if the rule of combination be relative multiplication, though the same ensemble does not possess even the group property if the rule of combination be logical addition or logical multiplication.

7. The earlier papers upon the strain of a gravitating elastic sphere were based upon a defective physical theory, in that the stress-strain relations were assumed to hold for a fixed element of space rather than for a definite element of matter. This defect was pointed out by Professor Hoskins in 1910 (*Transactions*, volume 11, page 504) and by A. E. H. Love in 1911 (Some Problems of Geodynamics, page 89). An amended theory was stated by Love and applied to certain problems including that of the tides in a gravitating sphere of uniform density. The present paper is based upon a theory in essential agreement with that of Love but formulated in a somewhat

more general manner. Professor Hoskins gives two applications: (1) to the problem of the strain of a gravitating compressible sphere of uniform density; and (2) to that of a gravitating incompressible sphere composed of a shell and nucleus of different densities. Both solutions follow the lines of the analysis employed in two former papers (*Transactions*, volume 11, page 203 and page 494). The same method can be extended to more complex cases such as that of a sphere composed of several shells and a nucleus, all of different densities and rigidities, such extensions being merely a matter of algebraic labor.

8. The complete existential theory of Dr. Bernstein's set of four postulates for Boolean algebras is shown by Mr. Taylor to consist of six propositions of non-existence and ten propositions of existence. The non-existences are expressed by the proposition $\Sigma^{-1} \supset \Sigma^{34}$. The propositions of existence are proved by the exhibition of ten systems Σ: two examples for K singular, six examples for K dual, and two examples for K triple. The postulates are shown to be completely independent in the sense defined by Professor E. H. Moore if the minimum number of elements of the class K be postulated as three.

9. Professor Blichfeldt proves that:

(*a*) No primitive linear group can contain a transformation of order $k > 5$ whose "multipliers" (characteristic roots) have only two values. For instance, no primitive group in three variables can contain the transformation $x_1 = x_1'$, $x_2 = \epsilon x_2'$, $x_3 = \epsilon x_3'$, whose multipliers 1, ϵ, ϵ ($\epsilon \neq 1$) are kth roots of unity; $k > 5$.

(*b*) If a primitive group G, say in n variables, contains a transformation S of order k, where $6 > k > 3$, having just two distinct multipliers, then these multipliers are each repeated $\frac{1}{2}n$ times. Moreover, G contains an invariant subgroup K. This is of monomial type and order 2^p when $k = 4$. If $k = 5$, then K is the direct product of say q groups of order 60φ and icosahedral type, and n must be divisible by 2^q.

W. A. MANNING,
Secretary of the Section.

THE TENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE tenth regular meeting of the Southwestern Section was held at the University of Kansas, at Lawrence, on Saturday, December 2, 1916. About thirty persons were in attendance, among them the following twenty-three members:

Professor C. H. Ashton, Professor A. A. Bennett, Professor Henry Blumberg, Professor W. C. Brenke, Professor P. J. Daniell, Professor E. W. Davis, Professor I. M. DeLong, Professor E. P. R. Duval, Professor W. H. Garrett, Professor W. A. Harshbarger, Professor E. R. Hedrick, Professor O. D. Kellogg, Professor S. Lefschetz, Dr. G. H. Light, Mr. W. A. Luby, Professor U. G. Mitchell, Professor Mary W. Newson, Professor S. W. Reaves, Dr. Paul R. Rider, Professor W. H. Roever, Mr. L. L. Steimley, Professor E. B. Stouffer, and Professor J. N. Van der Vries.

Attending members were very hospitably entertained at a smoker on the evening before the meeting, and at lunch between the sessions, at the University Club. It was decided that the next meeting of the Section should be held at the University of Oklahoma, at Norman, on December 1, 1917. The following programme committee was appointed: Professors S. W. Reaves (chairman), Henry Blumberg, O. D. Kellogg (secretary).

The following papers were presented:

(1) Professor A. A. BENNETT: "Newton's method in general analysis."

(2) Professor S. LEFSCHETZ: "Double integrals of the second kind belonging to an algebraic surface."

(3) Professor S. LEFSCHETZ: "On the residues of double integrals belonging to an algebraic surface."

(4) Dr. P. R. RIDER: "A note on discontinuous functions in the calculus of variations."

(5) Professor W. H. ROEVER: "Series for computing the roots of the equation $x + p \tan x = 0$."

(6) Professor E. B. STOUFFER: "On the calculation of invariants and covariants of linear homogeneous differential equations."

(7) Professor P. J. DANIELL: "The Lebesgue-Stieltjes integral."

(8) Professor HENRY BLUMBERG: "An example of a discontinuous function with a certain remarkable property."

(9) Dr. G. H. LIGHT: "Note on the relation between the focal points of an extremal when both end points are free."

(10) Professor G. C. EVANS: "An extension of Hadamard's formula for a linear functional."

(11) Professor H. C. GOSSARD: "On the relations between the faces and edges of a tetrahedron."

(12) Professor HENRY BLUMBERG: "Convex functions."

(13) Professor E. R. HEDRICK: "Functions that are nearly analytic."

Professor Blumberg's first paper and Professor Gossard's paper were read by title. Abstracts of the papers follow.

1. In this paper, Professor Bennett considers the formal character of Newton's method of approximating to a root of an equation, whether algebraic or transcendental. By generalizations of the notions sum, unknown quantity, coefficient, product, absolute value, etc., Newton's method is shown to apply also in a general field whose restrictions are considered explicitly. In particular, application is made to certain types of non-linear integral equations. The paper appeared in the *Proceedings* of the National Academy of Sciences, October, 1916.

2. The plan followed by E. Picard in his development of the theory of double integrals of the second kind is somewhat as follows: (*a*) reduction, largely by algebraic processes, of these integrals to a normal form; (*b*) development of the theory of integrals of total differentials of the third kind; (*c*) application of (*b*) to a special class of double integrals of the second kind; (*d*) investigation of the periods of integrals in the normal form; (*e*) enumeration of integrals of the second kind. In this note, Professor Lefschetz shows that this theory can be much simplified by starting with *d*. From this *a* and *b* then follow easily, and thus most of the algebraic work found in Picard's presentation is avoided.

3. A double integral belonging to an algebraic surface F may have cyclic or polar residues. It is shown in Professor

Lefschetz's second paper that the latter have the following two properties: (*a*) those with respect to points on the same algebraic curve have a zero sum; (*b*) those with respect to a given ordinary point of F also have a zero sum. The first follows at once from a well known theorem, while by means of quadratic transformations the second is reducible to a proposition given by Poincaré. Conversely, given a set of numbers corresponding in a suitable way to an assigned set of algebraic curves of F and satisfying the conditions a and b above, there is a double integral belonging to F of which they are the polar residues. No special conditions have to be satisfied by cyclic residues. This note will appear in the *Quarterly Journal of Pure and Applied Mathematics.*

4. In this paper, Dr. Rider gives the corner conditions and the Carathéodory Ω-function for discontinuous solutions in the calculus of variations for the form of the problem considered by Bliss in several papers, and for a new form of the space problem discussed by the author himself in a paper to be published in the *American Journal of Mathematics.*

5. For computing the roots x_n $(n = 0, 1, 2, \cdots)$ of the equation $x + p \tan x = 0$, Professor Roever gives two types of series. The first is a power series in p, the coefficients in the series for x_n being polynomials in the reciprocals of the number $t_n = (2n + 1)\pi/2$. The second type is a power series in $p + 1$, the coefficients being polynomials in the reciprocals of the numbers r_n, or the nth solution of the equation $x - \tan p = 0$. As $r_0 = 0$, a special formula is needed for the root x_0 in the second type of series.

6. Consider the system of linear homogeneous differential equations

$$(A) \qquad y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0 \quad (i = 1, 2, \cdots, n),$$

where p_{ik} and q_{ik} are functions of the independent variable x. In a paper presented to the Society at Chicago in April, 1915, Professor Stouffer calculated the complete system of semi-invariants of (A). In the present paper, he obtains the complete system of semi-covariants by the use of certain operators. He also sets up the systems of partial differential

equations which must be satisfied by any invariant or covariant of (A). These equations are expressed in terms of the seminvariants and semi-covariants. It is thus easy to test whether an expression in the seminvariants and semi-covariants which may arise in the geometry of the system is also an invariant or covariant.

7. The Lebesgue-Stieltjes integral is defined by Professor Daniell in this paper by means of sets of points in a way similar to that employed in defining the Lebesgue integral. It possesses the Lebesgue property that the limit of the integral is the integral of the limit. Like the Stieltjes integral, it is discontinuous, and the "measure" of even a single point may be finite. It is therefore suitable for use in physics, particularly electricity, where charge (or mass) may be used as a "measure" of a set of points. Most of the fundamental properties of Lebesgue integrals are shown to be satisfied by it. The paper will be offered to the *Transactions*.

8. Professor Blumberg shows with the aid of Zermelo's Wohlordnungssatz that there exist real, one-valued, discontinuous functions $f(x)$, defined in the linear continuum, such that if P is a given perfect set, then $f(x)$ is discontinuous with respect to P at every point of P.

9. In the *Mathematische Annalen*, volume 75, Rasmadse has considered the case of one fixed and one variable end point. He found that the focal point t_2'' comes before the focal point t_1'' if the integral $J = \int_{t_1}^{t_2} F(x, y, x', y')$ is a minimum. Dr. Light shows that t_2'' and t_1'' must coincide in the general case in which both end points are free.

10. In a recent paper in the BULLETIN (November, 1916) Dr. Fischer gives an interesting formula for a linear functional $T[\varphi]$ which has continuity merely of order k in its argument $\varphi(x)$, instead of order zero. Hadamard's form for a linear functional, which is itself slightly more general than the others, admits, according to Professor Evans, this extension in a very simple form. If T, depending on $\varphi(x)$ in the interval (a, b), has continuity merely of order k, it may be written: $T[\varphi(x)] = \lim_{n=\infty} \int_{a-\epsilon}^{b+\epsilon} \varphi(x)\Psi(x, n)dx$, where $\varphi(x)$ is extended

continuously with its first k derivatives beyond the interval (a, b) to the interval $(a - \epsilon, b + \epsilon)$, ϵ being a quantity arbitrarily small, but positive. For $\Psi(u, n)$ may be taken such a function as $\sqrt{n/\pi}T[\{1 - (u - x)^2\}^n]$. These results will probably be published in the Cambridge Colloquium Lectures.

11. By expressing the absolute in terms of the faces of a tetrahedron and also in terms of the edges, both equations being written in line coordinates, and then equating coefficients, Professor Gossard obtains (*a*) six equations of the type $(3V/2)^2e_{12}^2 = A_{00}A_{33} - A_{03}^2$; (*b*) twelve of the type $(3V/2)^4e_{12}^2e_{20}^2 = (A_{03}A_{13} - A_{01}A_{33})^2 + 4A_{33}(3V/2)^4$; (*c*) three of the type $9V^2[(e_{01}^2 + e_{23}^2) - (e_{02}^2 + e_{13}^2)] = 16(A_{01}A_{23} - A_{02}A_{13})$, where V is the volume, A_{ij} the product of the areas of the faces opposite the vertices $e_i = 0$ and $e_j = 0$ by the cosine of the angle between the two faces, and where e_{ij} is the length of the edge from the vertex $e_i = 0$ to the vertex $e_j = 0$.

12. The real, one-valued function $f(x)$ of the real variable x is (according to Jensen) said to be "convex" in the interval (a, b), where it is defined if, for every pair of values x_1, x_2 of x, the inequality

$$f\left[\frac{x_1 + x_2}{2}\right] \leqq \frac{f(x_1) + f(x_2)}{2}$$

holds. Similarly, the function $f(x, y)$, defined in the square S, is said to be convex in S, if for every pair of points (x_1, y_1), (x_2, y_2) in S, the inequality

$$f\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right] \leqq \frac{f(x_1, y_1) + f(x_2, y_2)}{2}$$

holds. Professor Blumberg shows that if the convex function $f(x)$ has at least one discontinuity between a and b, it is necessarily non-measurable in the Lebesgue sense. Furthermore, if the convex function $f(x, y)$, defined in the square S, is such that the set of functional values of $f(x, y)$ on the boundary of S has a finite upper bound, then $f(x, y)$ is continuous at every interior point of S. This result remains true if for "square" we substitute "closed Jordan curve," and, in fact, more general curves may be allowed. Generalizations to n dimensions are given.

13. In this paper Professor Hedrick describes a method of characterizing functions of a complex variable that are not analytic, but that differ only slightly from analytic functions. For example, the function $\psi(z) = f(z) + \epsilon\varphi(z)$, where $f(z)$ is analytic, and ϵ is a real number, will differ but little from $f(z)$, if ϵ is chosen small, at all points where $|\varphi(z)|$ is finite. The Riemann surface for such functions may be studied by means of its relation to that for $f(z)$. A series of examples of this type and of more general types is given.

O. D. KELLOGG,
Secretary of the Section.

THE RELATIONS OF MATHEMATICS TO THE NATURAL SCIENCES.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1916.

BY PRESIDENT E. W. BROWN.

THE duty enjoined on the President of the American Mathematical Society of delivering an address at the close of his term of office is at once an opportunity and a danger. It is one of the rare occasions when he is able to discuss matters which are unsuited for a memoir and when it is proper to try to take a somewhat broader view of his subject than is suggested in investigations designed to elucidate some special part of it. In so doing, he necessarily must look into the future and attempt to foresee it through such indications of the present as may seem significant; and the danger of becoming a false prophet or of raising an unnecessary alarm is unattractive to anyone, least of all to those who have the lifelong habit of feeling their way into the unknown by roads slowly constructed and securely laid. I am willing to run this risk because I believe that there are certain matters connected with the future of mathematical science which need fuller consideration than they have received of late years. In discussing them, one must necessarily tread on debatable ground. It is, however, a happy custom to regard the matter contained in a presidential address not as an official presen-

tation of the views of the Society but solely as the personal opinion of the one member appointed to deliver it. Such suggestions as I have to make in connection with my topic "The Relations of Mathematics to the Natural Sciences" have no claim to novelty or originality. They are partly the natural result of association with those who work in more or less closely related fields and partly of a long connection with a task which has led in directions away from those usually followed by the modern mathematician. Before this discussion, however, a brief summary of the principal events of the past two years may not be out of place.

The summer meeting of the Society in Cambridge, as you are aware, was made the occasion for a celebration of our twenty-fifth anniversary. Originally started in 1888 as a meeting of those interested in mathematics in New York and its vicinity, the Society emerged three years later in the form of a national organization with a BULLETIN containing a record of the activities of its members and with some limited publication of their work and ideas. Nine years later, the growth of research in America, unquestionably assisted by the opportunities afforded by the Society, caused a need of further facilities for publication and the *Transactions* was started. Both journals have grown beyond the sizes originally contemplated and we may hope or fear that a new medium for publication may be required in the not distant future. The original meeting place in New York City has been quite insufficient to supply our members with opportunities to present papers and to engage in discussions; one after another, sections have been founded in various parts of the country which hold sessions at regular intervals and which assist actively in contributing to the growth of the subject and the welfare of the society. Our library numbers over 5,000 volumes, besides a large collection of unbound dissertations. And finally the offices of the Society and most of its records have survived the fire which seems to be a necessary preliminary to permanent stability.

This growth has lately caused a serious problem to arise in connection with the extent of the activities of the Society. Many of those who on account of professional duties or for other causes are unable to follow or lead far along the highroads of research, have felt that there are many byways in mathematics which are of interest and which would stimulate

thought. They felt too that some centralization of efforts to improve the teaching of mathematics was necessary. The Society had hitherto confined its main activities to research and it became a grave question whether it should undertake new duties and responsibilities in addition to the already heavy burdens laid on certain of its officers. The difficulty has been happily solved by the formation of the Mathematical Association of America with assumption of responsibility for the *American Mathematical Monthly* and its use as the official journal of the Association. This result has been brought about by the cordial cooperation of everyone concerned. They have desired not only that the interests of the American Mathematical Society should not suffer but that the new Association should be so organized and conducted as to assist in strengthening and coordinating all forms of mathematical activity throughout the country. Our best wishes go out to our young and already vigorous offspring.

In outside activities, the Society during the last two years has appointed representatives to the general Committee on the classification of technical literature, to the Committee on standards for graphic representation and to the Naval Advisory Board. In spite of some misconception in the minds of those who were responsible for the invitation to the last named, as to the functions of mathematicians, both the Society and the Board are to be congratulated on the work of our two representatives.

Perhaps the most important outside movement affecting the chief objects of the Society is that initiated by the National Academy of Sciences for the organization and development of research. Originally started with the object of assisting in the defence of the country, it enlarged its functions when it was seen that much would be gained through cooperation and mutual assistance not only by investigators themselves but by all those who need and employ investigators in technical and manufacturing processes. This is therefore a movement in the interests of the whole community. The Society has already, through the Council, expressed its desire to cooperate. We may hope that there will result some benefits for our science and that we shall be able to furnish in turn some assistance toward the solution of the problems presented by other sciences.

With this record of achievement in the past it is but natural

that we should look forward to the future and note whether, besides promise of progress and success, there are indications of danger against which measures of protection or alleviation are necessary.

Mathematics may be treated as a science or art which holds within itself all that is necessary for its own development. It may also be treated as an aid for the development of the natural sciences and for all forms of investigation which admit of deduction from exact statements. In the one, an advance in knowledge of the subject itself by examination of the interrelations of a set of axioms is sought; in the other, it is used to deduce phenomena of nature as the consequences of physical laws which have themselves been deduced by observation or experiment. In the earlier history of the subject, the difference between the two points of view was but little recognized. The axioms themselves were mainly suggested by natural phenomena and the developments were principally those needed in the elucidation of the phenomena. As time went on, mathematics was allowed to progress in its own way, unhindered by any necessity for present or future applications. This freedom from a somewhat entangling alliance has resulted in a marvellous structure of thought. The fundamental bases of the subject have been clarified and organized, the modes of reasoning have become subject to careful scrutiny and every effort has been directed to make sure that the structure built on these lines shall be without blemish.

There is, however, a reverse side to the picture. While every other product of human thought demands aid from outside, modern pure mathematics stands practically alone. Its laws, its logic, what it seeks and what it finds have no necessary relation to externals in the sense of dependence. And this independence has been achieved by no ordinary labor and by the exercise of many minds possessing gifts of insight in the highest degree. Yet one cannot help asking whether it is for the best interests of the subject that it should continue in this isolation. An external stimulus seems to be necessary, at least from time to time, to produce the best elements of growth in most forms of human activity. Is the continued development of pure mathematics an exception to the rule? One may argue that isolation produces a pure strain, but is it not also true that an occasional crossing of the breed is necessary to prevent the species from running itself

out? I do not wish to imply that this is a real analogy; whether it is so or not, it can at least serve as an illustration of the manner in which the separation may affect mathematics. If it be admitted that future injury is likely to result, it is obvious that we should search for methods to alter the condition; if the contrary is true, I can still fall back upon the demand that other sciences make for assistance.

At the outset one must admit that the separation is not altogether confined to the relations of mathematics to the natural sciences, it has been growing even between its different branches, not so much perhaps in essentials as in the fact that there are only a few amongst us who can follow except in a dim way or with much labor, the larger part of the advances which are being published so frequently. This is partly due to the considerable body of knowledge which has accumulated: time and opportunity are lacking to assimilate even a small portion of it. And there are other contributing causes which will occur to you. Over one of them, of minor importance perhaps but in which some improvement may be possible, I shall delay for a moment.

The symbolism employed to convey ideas in mathematics is unquestionably good in its general outlines and serves the purposes of the study well. It is brief, it admits of considerable variety, and it is easily adapted to most of the needs of the mathematician. Unfortunately, few standards have been kept. While the general principles by which the notation is to be used are little changed, the widest variations in the meanings of different sets of symbols have been allowed to creep in; and conversely, the same sets of ideas have been frequently represented by a variety of symbols. In my own subject of celestial mechanics, this variety continually halts one's thoughts although the mathematical ideas are comparatively simple; in fact the symbols rarely represent anything else than the most elementary operators and quantities which are either fixed or have continuous variation. But the difficulty is far greater where the ideas themselves are complex, where a single symbol may imply numerous properties and where different authors adopt different meanings to be attached to the same symbol. The perfectly simple triad of marks "(1)," for example, may represent a number, an operator, a group, a function, an axiom, a convention—and any one of these may have special properties attached or particular limitations

imposed. And when to this are added many other symbols with similar extensions of meaning, one difficulty of mastering a memoir of average length becomes sufficiently obvious. The language must first be learned, and a new language in general confronts the reader when he turns to some memoir by a different author, even on the same subject.

Is there a remedy for this condition? Can mathematical notation be standardized to some extent and if so, can it be done in such a manner that it will meet with general acceptance and use? It is perhaps a Herculean task but I am not at all sure that a concerted effort would not result in some improvement.

With what material have we to deal? On the one side there are the four double fonts of the Roman, Italic, Greek, and German alphabets, one or two fonts of the Arabic numbers and a few well-known symbols, the types for which exist in any good printer's composing room. On the other side there is a considerable body of ideas to be represented, although the number of those most frequently used is not very great. Suppose, by way of illustration, that some very general conventions were agreed upon: for example, that operators should be denoted by capital letters, except in the few cases like those of differentiation or integration, where the notation is already standardized; continuously varying quantities by certain groups of lower case letters, discontinuous quantities like those in our number system by other groups, quantities remaining fixed by still other groups. Continuing with this idea we should represent those properties or limitations which are of frequent occurrence by certain combinations of marks and letters and it would be advisable to agree that the older signs, those of the four chief arithmetical operations for instance, shall retain their original meanings if unattended by any distinguishing mark. Any such plan should depart as little as possible from the schemes of notation which have been most generally used. One result undoubtedly would be some loss of the brevity which now characterizes the mathematical symbolism of much of the published work. I doubt, however, whether this would be an evil, in fact, it might be an advantage; the reader would soon become familiar with many groups of symbols and would read their meaning without effort, just as he recognizes a printed word not by the separate letters but by the appearance of the group of letters which form the

word. After all, brevity is not the soul of mathematics. If a scheme with such broad outlines is not feasible, it might still be possible to apply the method separately to divisions of mathematics. But this idea is less attractive and presents additional difficulties in the overlapping of different divisions. Most of us would prefer, I think, to learn one language with many words, than several different languages even though their vocabularies were much smaller.

It may be objected that any standardization of symbolism may seriously interfere with the freedom of the investigator and so limit his output. One may ask in reply whether past liberty has not already degenerated into license and whether some regulation will not confer benefits on mathematics similar to those we acknowledge in the framing of laws for the welfare of the community. If the circulation of the output be increased we can well afford some decrease in the quantity.

It is a much more simple matter to suggest a method for bringing any such scheme or schemes into operation if once agreed on by an authoritative committee. After proper and frequent publication, there is no need for any kind of compulsion; a writer, using a settled scheme has merely to say so. In this way it would either come into general use or die a natural death without having caused any particular harm. There is another consideration in this connection. We must look forward to the time when our turn will come to publish a revised encyclopædia of mathematical sciences; if standards of notation are then available, one can hardly overemphasize their value in making a summary of the work of the past.

In the meantime some assistance can be given by such aids as tables of notation and separate summaries attached to each paper published, even at the cost of artistic appearance; this has already become a feature of some journals in other subjects. And there are probably other devices which might usefully be employed to assist the reader.

These are, however, mechanical aids and only touch on the fringe of the question which is my principal theme—the isolation of mathematics from the natural sciences. There is, I must confess, at present little prospect of the cure generally proposed, namely that mathematicians turn their attention to the applications. And there is good reason why this is not acceptable, at least in the form usually stated. To make progress in any one direction requires a more considerable

knowledge of the physical laws and past achievements in that direction than most of those who are making advances in their own lines have time to acquire. They may do much by encouraging a certain proportion of the students who come under their influence to attack such problems. The welcome given by the mathematical journals in America to memoirs on the applications is a sufficient answer to those who would believe that the study of the latter is discouraged. In this connection it is noticeable that the pure science has a tendency to drive out the applied science—this has been peculiarly evident in England during the last quarter of a century—or to put the matter in another form, the cleavage between pure mathematics and the experimental sciences which use mathematics has tended to increase.

It is necessary here to distinguish between the study of the mathematical methods used in the applications and the study of the applications by mathematical methods. In the former, the foundation, the logic and the limitations which must be imposed on the processes adopted by applied mathematicians are considered; in the latter, it is new phenomena or the coordination of known phenomena which are sought. Poincaré, who produced much in both directions, obviously kept the distinction clear in his own mind, even when he mingled the two in a single memoir like his treatise on celestial mechanics. And his point of view in the former was not that of criticism but of justification of the methods used in the latter. Unfortunately, while a study of methods clarifies and expounds, it is not as a rule fertile in producing new advances. It is only when the mathematician leaves aside the particular applications of a method and studies the latter in all its bearings that he is able to obtain novelty.

The study of physical laws and phenomena by the methods of mathematics stands on an altogether different basis. Why is it that with calls in every direction for help so little is done? One hears blame cast on the mathematician for this state of affairs and perhaps to some extent there is justification for it. But I believe that the blame must be shared at least equally by the physicist or engineer who asks for the solution of his problems. The latter are generally presented in such a manner that the mathematician who is not thoroughly familiar with the subject cannot even make a beginning; often he is not given the necessary information about the physical cir-

cumstances which will enable him to formulate the problem. In its generality, it may be quite beyond the present state of mathematics to give a solution and yet if he is allowed to neglect certain factors and to modify others it may be possible to obtain the main facts needed. But he cannot be expected to know all these details. The more difficult the problem, the more necessary is a full and complete statement of everything connected with it. The physical laws which may be assumed and which cover the principal features, the modifying circumstances which may be left aside if the problem is too difficult, the range of numerical values which constitute the data, the range of numerical values which are needed in the solution—all these should be carefully set forth in any statement of the problem.

It is this work of formulation that is mainly contained in the treatises on applied mathematics so far as the older and more simple problems suggested by natural phenomena are concerned. They also give the principles which should be followed in all cases. Thus the knowledge of the applied mathematician, using the word in the old sense, is necessary as the intermediary between the experimentalist and the pure mathematician if one is to help the other. He must have a considerable knowledge of the phenomena and with it a sufficient knowledge of mathematics; neither can be neglected. It is the almost complete disappearance of this type of scholar from among us that is perhaps responsible more than anything else for the isolation of which I have spoken. The temptation either to proceed to experiment or to turn to purely mathematical researches has been too strong.

The need of a remedy is obvious and in searching for it we may perhaps have to consider a change in our system of regarding the doctor's degree with its original thesis as a necessary preliminary to every opening which leads to scientific work. It is frequently stated that the main difficulty in obtaining applied mathematicians is the difficulty of finding a suitable thesis for a student, by an instructor who has not himself investigated in those lines, and we have few here who have done so. This is unquestionably true. Under the circumstances, those organizations which need such men for their own purposes, whether they be government bureaus or commercial houses, must create positions for applied mathematicians with the same advantages that they give to those who

perform experimental researches. They must understand that the type of man needed must be trained for the purpose, that he cannot usually be solely a physicist or solely a mathematician but must have received a sufficient education in both directions. If the demand is clearly set forth, there is little doubt that the supply will be forthcoming.

The necessary link once supplied, the pure mathematician can render valuable services. But to do so, he will need to change to some extent the methods which he is accustomed to employ. The generality and completeness which he is accustomed to give to his researches are rarely useful in the applications. What is needed is the discovery of solutions which in a limited range can be reduced to numerical values without an excessive amount of labor. Singularities, for example, are usually excluded for one cause or another; places where the function is regular are those commonly needed. If the constants expressed by symbols are so numerous that the problem is intractable, some of them must be given numerical values; it will frequently happen that all needed information can be obtained from two or three well chosen numerical cases. And it is not only the material stored up in the past that will be used; new devices will continually be needed and I do not think that he will find the problems any less worthy of his best efforts than those to which he has been in the habit of directing his attention. The chief difference will perhaps consist in a limitation to work in definite directions instead of roaming in the field of discovery with freedom to investigate wherever the prospect seems to attract.

It follows that some organization of the different processes needed in the applications of mathematics has become necessary. The simple problems and a very few of the difficult ones have been solved and progress is halted in the face of a demand which is greater than ever before. Whether one considers questions of purely scientific interest like those which are presented in celestial mechanics or questions of practical importance, it is still true that there are few workers and even with their combined efforts they cannot achieve much. If we are to enlist the services of others, it is necessary that the work be divided in such a manner that contributions to its various parts can be made by those who have the proper training and qualifications. The first question suggested by what has been said is to ask what part can be taken by those who have a training in pure mathematics.

It is always safe but not always useful to restrict oneself to generalities. I am fully aware of the danger which attaches to the mention of specific problems; they have been solved, or they are insoluble with known methods, or they are not suitable for mathematical treatment. Nevertheless, I am going to mention briefly two or three which I believe cannot properly be placed in any one of these excluded classes. So far as I know, there is no published systematic attempt to treat them in such a manner as to furnish results in a form available for actual application. Too much emphasis can scarcely be laid on the last point if the researches of mathematicians are to be connected with the applications. Much of such work may be tedious and uninteresting but there is also much which demands the exercise of the best powers which an investigator may possess. He is also in the most favorable position to do it. He has usually looked at his subject from many directions before setting forth that one which is best suited for a theoretical exposition and he will know and be able to develop most easily that which is suitable for the applications.

Of the fundamental requisites which need systematic examination, perhaps that of most practical value is an inquiry into the symbolic forms and modes of expression which are best adapted for numerical calculation. On the one side there exists a wealth of material in the shape of formulas, theorems, and methods available for choice. On the other side the machinery to which these are to be adapted is limited. Practically, the only numerical operations which can be performed mentally are addition, with its counterpart subtraction, and those operations in which a knowledge of order of succession in our decimal number system can be of assistance. All other operations are performed by means of tables whether memorized like the multiplication table or printed. Mechanical and graphical devices to replace these are available but the only one which has complete accuracy is that which replaces the mental operations, namely, the adding machine. The number of printed tables at our disposal is so small that a classification of these containing an account of their contents and of the uses to which they may be put is not a serious undertaking as a first step. Finally, the various devices by which the labor of numerical computation may be diminished should be collected; this is more troublesome because many of them are the results of experience and either have not been

published or are only mentioned incidentally in connection with special investigations.

It is a much less simple matter to try to point out a method of procedure for the adaptation of symbolic forms so that full use may be made of this numerical machinery. Something might certainly be achieved by gathering together those forms in most common use in a given subject or group of subjects and making a study of them from this point of view. An example occurs in the case of Fourier series; a number of papers have been published during the present century on the methods for connecting a Fourier series with its numerical values although most of the attention given has been in the direction of analysis rather than synthesis except in the case of purely mechanical devices. One may perhaps look forward to the construction of a more general theory which shall have as its object the formulation of laws or rules which will serve in all cases. Adaptation to numerical computation and the performing of the computations themselves are by no means the simple matters which they are frequently assumed to be if they are to be efficiently done. And it is this efficiency which must be the goal.

The converse problem of reducing a series of numerical values which obviously obey some law to symbolic form is equally in need of systematic treatment. If the law is known or there are reasons for the assumption of a specified form, the method of least squares enables the unknown constants to be determined. But it is rare that one opens a series of memoirs containing investigations in the physical sciences without seeing diagrams for which no law is known; in many cases no attempt is made to find one. The author has perhaps drawn a smooth curve which he judges would most nearly represent the facts if the observations had been freed from the errors which are incidental to all experimental work. But progress is halted until the curve can be so expressed that the relation of the physical phenomena that it represents to other phenomena or to known physical laws can be obtained. The best method of doing this is usually through a symbolic formula. The mere fact that functions with limited variation (and the known phenomena possess this property) can be represented by a Fourier series is usually of little or no assistance. Even if it be so represented it conveys almost no further information unless the phenomena themselves are of a periodic nature.

The Fourier series might perhaps be used as a first step in the investigation but there arises the difficult and practically unsolved problem of finding the function represented by the Fourier series, as was pointed out in a recent address. Some more direct and practical method is needed. In certain cases a mode of procedure can be indicated. When the curve appears to resemble the graph of some known function, it should be possible to devise methods of approximation by which additions to or changes in the function can be obtained, the constants present being then determined by the method of least squares. Transformations of functions so obtained involve the study of forms numerically equivalent within a given range to a given degree of accuracy.

The case of periodic or quasi-periodic functions deserves special mention. Light-curves of many stars are now being obtained with considerable accuracy. Some of the curves differ from a simple sine-curve so greatly that the representation by Fourier series will probably involve the presence of a large number of terms with sensible coefficients. Such forms will probably require transformation into other periodic functions if any clue is to be obtained to the laws which govern these changes. Our own sun is no exception. We obviously have to deal with forces which do not produce regular periodicity, but until some analytic forms can be obtained which will represent the curves with sufficient brevity and accuracy there is little hope of unravelling their mysteries.

I have mentioned these problems particularly because they lack the definiteness in the results demanded which characterizes most of the usual problems of mathematics. I do not think they belong to the subject less for that reason. Mathematics certainly can lay claim among its other properties to be a method of organizing ideas, and it has effectively done so in the past. The result of tossing a coin into the air is, owing to our lack of ability to work out all the consequences of the laws of mechanics, quite as indefinite, and yet mathematics has built up the solid and definite set of rules which we know as the theory of probabilities—a theory which is in practical use in every insurance company and in all scientific work: it is difficult to imagine what modern astronomy would be today without the method of least squares. Such problems as I have suggested may possibly have only temporary value but if in their solution some order is produced from chaos, a

contribution of practical use, at least in our time, will have been made.

Perhaps the set of mathematical problems most important for the applied sciences are those included in obtaining numerical solutions of differential equations. In many cases the ordinary methods such as finding exact solutions in terms of known functions or by means of series fail. The failure in the latter case is not a mathematical failure, but a practical one, owing to the large amount of numerical computation needed. Where it has to be undertaken, the particular method of analytical continuation of series, known as mechanical quadratures, is adopted. This certainly achieves the object in view but usually with a lack of real efficiency which cannot fail to strike anyone who attempts to use it to any extent. And it naturally becomes the more troublesome whenever the curve has any special feature such as near approach to a singularity. Questions which seem unimportant in theory sometimes loom large in calculation. For example, how many places of decimals at the start and throughout the work are needed to secure a given standard of accuracy? Is the same degree of accuracy necessary throughout? How large may each arc be taken? And so on.

If one examines textbooks or treatises for information, it is difficult to find anything of real use; in fact, the method as applied to the solution of, for example, a pair of ordinary simultaneous differential equations, I have not seen mentioned. In the form given in the investigations where it has been used, it is doubtless capable of great improvement. We may further ask: under what forms can it be used so that each arc shall be as long as possible with the same degree of accuracy? Is it necessary to confine it to power series and can it not be developed for computation in series of other functions so that, in any special case, one shall have a choice which can be made according to circumstances laid down in advance? Unquestionably there is material at hand to answer some of these questions if it could be worked up into a form simple enough for a person not possessing an intricate knowledge of differential equations and the properties of various modes of expansions to use. I hesitate in mentioning solutions of partial differential equations in this connection on account of the many inherent difficulties. Nevertheless the new problems arising in aviation and the older set connected with the motions

of our atmosphere and the prediction of the weather perhaps transcend in importance all others.

These problems are fundamental in the applications of mathematics. The expert knowledge and insight of the student of pure mathematics would seem to point him out as the one most likely to achieve at least partial successs in dealing with them. At the same time, little or no knowledge of physical processes and results is required. But if he happens to possess special information in any one of the natural sciences, I have little doubt that he can find a field giving a stimulus for new work and an opportunity to connect his subject with the outside world. Even biology is now demanding help in certain directions, somewhat reluctantly it is true, but none the less surely. Even if mathematics cannot always assist in solving problems, it can still help through its advantages as a brief and clear mode of description where processes are complicated and interdependent. But whatever is done in any direction, the work will be of little practical value unless it satisfies the final test, namely, that the form in which it is left is such that it may be applied without a full knowledge of the processes by which it is reached.

It is perhaps easy for one who has spent half his life in an undertaking which must always have numerical results as the final test, to exaggerate the importance of all work of this nature. This particular problem, it is true, has a variety which is not common to many such undertakings. Perhaps the most unexpected result has been that nearly all the portions which can be said to possess some small degree of novelty have arisen from the necessity of obtaining formulas which should give numerical results more easily than the older developments. And it has not infrequently happened that the actual computation of a particular case has indicated immediately the method which should be followed in the general case. This fact may not be a recommendation to the person who has the desire to use his powers to their fullest extent, yet for one who has a definite end in view and who wishes to reach the goal by the shortest route it may appeal as having certain advantages. Moreover, the time spent in actual computation is usually not a large fraction of that necessary for the whole work; if there are many computations of the same nature, each one may take on the average only a fifth or a tenth of the

time that an isolated case would require. The construction of numerical tables is a striking example of this as soon as the work has been properly organized.

Much of the mathematics used in the applications is repulsive to the pure mathematician, and properly so. It lacks the form and symmetry which so often assist in showing the ideas which are in process of development. There is a good reason for this. For efficient numerical application, many properties possessed by the initial formula and particularly that of symmetry should be used to assist; in the process of transformation they must necessarily disappear. The limitations, too, which are often of greater interest, are usually so much broader in the mathematics than in the applications demanded that there is no need to examine them. Or if they are reached, the numerical work at once warns of their approach and the formula or method is changed. Unfortunately, too, the very devices which constitute a good formula or by which a solution is reached often admit of such simple mathematical description that the air of mystery disappears and there is little external evidence of any special ingenuity. Nevertheless there is a degree of mental satisfaction in the achievement of concrete results which is a sufficient reward to those who work for them.

In this attempt to emphasize the need for more attention to the applications of mathematics, it is hardly necessary for me to deny a possible implication that the further study of the pure science is unprofitable. On the contrary, there seems to be more need for it than ever. But I believe that this need demands studies which shall take directions more immediately helpful to other sciences, not only for their assistance but for the sake of the subject itself. We cannot leave such matters to the chances of the future; some definite effort is required to work towards a goal which has been neglected in the past. It demands, it is true, limitations on one's freedom to follow attractive openings, it may require tedious and uninteresting developments, but these will not be deterrents to anyone who has previously attempted to advance knowledge without hope of external reward. Moreover, the natural bent of a nation which has been compelled by its environment to regard most of its affairs from a thoroughly practical point of view, gives promise of success. An incentive and the training once supplied, there seems to be no reason why a school of applied

mathematics of the first rank should not flourish on this continent.

Finally, any discussion which has investigation as its theme is incomplete if it leaves out of consideration the events which are taking place among the older nations. Although as a body and as individuals we have perhaps been less affected than others engaged in intellectual pursuits, nevertheless, these events cannot fail now and more so in the future to modify —perhaps to change entirely—our outlook. While the need for scientific research has been emphasized and brought forward more strongly than ever before, and in this movement we hope to share, yet it is to be remembered that a practical end has been visible throughout. Little stress has been laid on its purely intellectual side and we, in common with others engaged in research in pure science, may fear neglect and increasing lack of support. For a time at least, the problems of reconstruction and adaptation to new conditions will demand of the older nations most of the time and energy which has in the past been allowed to run in less practical routes. It is true that here we are happily removed from the devastation which has overtaken Europe, but the lack of stimulus from abroad cannot fail to have its effect. We need at this time to take stock of the resources of ourselves and of our Society, both mental and physical, to find whether they are sufficient for what may come and more than all to conserve the heritage which we have received from the old world so that it may later be returned alive and productive.

The possibility of great changes coming rapidly and with little warning must be faced. We cannot therefore be content to sit back and await events. The future of research in pure science is in danger as never before and the less practical the science the greater the danger. New economic and social forces are at work in every direction and may call many of us to activities which may stop temporarily and perhaps permanently the work and thought which have in the past most fully occupied us. Two generations of steady and continuous development have tended to make us settle in grooves of thought and habits of mind in which great upheavals have no place. To such changes at least a proper mental attitude is necessary. Action may not be needed now but preparation for action is essential and the chances of the future should be in the mind of every individual if the right course is to be pursued

when the time for action comes. Even at the present moment we can no longer regard the Society merely as one to forward mathematical research in America; the wider interests of the subject have now to be considered, since we form the one large national organization devoted to research in mathematics which can continue its work unhindered by more pressing claims. In these wider interests our opportunities for service may come: the readiness and ability that we show in a realization and fulfilment of them will be the chief measure of our success in the past and promise for the future.

YALE UNIVERSITY,
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A THEOREM ON THE CURVES DESCRIBED BY A SPHERICAL PENDULUM.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, December 27, 1916.)

1. IN what follows I shall restrict myself to the case considered by Greenhill,* in which the pendulum bob reaches (but does not go above) the horizontal plane of suspension with a non-vanishing horizontal velocity.†

When this assumption is made, there exists an interesting relation between the curve described by the bob and a certain quartic in space. It is the purpose of this note to establish this simple relation.

2. Let r_0 be the distance of the bob from the vertical axis when its horizontal projection crosses the x-axis; $A + 2\pi$ the horizontal angle between two consecutive positions in the plane of suspension; $4K$ the real period in the Jacobian elliptic functions appearing in the solution; $v = t/2w_1$, where w_1 is the real half-period in the Weierstrassian form, and t the time; a, b, c $(a > b > c)$ the real roots of the cubic in the general solution; $\mu = \sqrt{(l^2 + a^2)/(l^2 - a^2)}$, where l is the radius of

* Greenhill, Les Fonctions elliptiques et leurs Applications, chap. III. Appell, Traité de Mécanique rationnelle, t. 1, p. 494. Tannery et Molk, Eléments de la Théorie des Fonctions elliptiques, t. 4, p. 188.

† The statement by L. Lévy, on page 161 of his Fonctions elliptiques, that in this case x, y, z are doubly periodic functions of the time is not correct.

the sphere. As in this case $b = 0$, we find, according to Tannery, loc. cit., the positive sense of the z-axis being downward:

$$(1)\quad x = r_0[\cos(A+\pi)v \cdot \mathrm{cn}(2Kv) - \mu \sin(A+\pi)v \cdot \mathrm{sn}(2Kv)\mathrm{dn}(2Kv)],$$

$$(2)\quad y = r_0[\sin(A+\pi)v \cdot \mathrm{cn}(2Kv) + \mu \cos(A+\pi)v \cdot \mathrm{sn}(2Kv)\mathrm{dn}(2Kv)],$$

$$(3)\quad z = a\mathrm{cn}^2(2Kv),$$

for the coordinates of the bob in parametric form.

From (1) and (2) it is easily seen that

$$(4)\qquad x\cos\phi + y\sin\phi = r_0\,\mathrm{cn}(2Kv),$$

$$(5)\qquad -x\sin\phi + y\cos\phi = \mu r_0\mathrm{sn}(2Kv)\mathrm{dn}(2Kv),$$

where $\phi = (A+\pi)v = (A+\pi/2w_1)t$ is an angle proportional to the time.

If we now put

$$(6)\qquad x' = r_0\mathrm{cn}(2Kv),$$

$$(7)\qquad y' = \mu r_0\mathrm{sn}(2Kv)\mathrm{dn}(2Kv),$$

$$(8)\qquad z' = a\mathrm{cn}^2(2Kv);$$

it is seen that the locus of the points (x', y', z') is a quartic in space. For each of the functions $\mathrm{sn}(2Kv)$, $\mathrm{cn}(2Kv)$, $\mathrm{dn}(2Kv)$, has $iK'/2K$ and $(iK'+2K)/2K$ as the only two poles within the period parallelogram, so that any linear combination

$$\alpha x' + \beta y' + \gamma z' + \delta$$

is an elliptic function with four poles (counting multiplicities properly), and consequently four zeros within the period parallelogram. The curve is therefore cut in four points by every plane and is a quartic.

The coordinates of a point $P(x, y, z)$ of the pendulum curve and the corresponding point $P'(x', y', z')$ of the quartic are connected by the formulas

$$(9)\qquad \begin{aligned} x' &= x\cos\phi + y\sin\phi, \\ y' &= -x\sin\phi + y\cos\phi, \\ z' &= z. \end{aligned}$$

Hence, as x, y, z are functions of the time, the point P' is obtained by rotating the point P (in the negative sense) about the vertical axis (z-axis), through an angle which is proportional to the time t, with $-[(A+\pi)/2w_1]$ as a proportionality factor. Hence the

THEOREM: *If the points P of the curve of a spherical pendulum motion, which just reaches the horizontal plane of suspension with a non-vanishing horizontal velocity component, are rotated in the negative sense about the vertical axis through angles proportional to the times (with a definite proportionality factor) associated with those points, the locus of the rotated positions of the P's is a quartic in space of the first kind.*

4. The elimination of v between (6) and (8) leads to the equation of the parabolic cylinder

$$ax'^2 - r_0^2 z' = 0. \tag{10}$$

It touches the plane of suspension along the y-axis, it is below this plane, and is symmetrical with respect to the yz-plane. The elimination of v between (7) and (8) gives the equation

$$a^2y'^2 + \mu^2 r_0^2 k^2 (a-z)^2 - \mu^2 r_0^2 a(a-z) = 0, \tag{11}$$

which represents an elliptical cylinder. It cuts the z-axis at a distance a below and at a distance $a[(k^2-1)/k^2]$ above the origin. The quartic is now obtained as the intersection of these two cylinders and consists of only one real branch. The horizontal projection of this curve is a plane quartic of deficiency 1, and with the equation

$$\mu^2 k^2 x'^4 + \mu^2 r_0^2 (1-2k^2) x'^2 + r_0 y'^2 - \mu^2 r_0^4 (1-k^2) = 0. \tag{12}$$

The infinite point of the y-axis is an isolated point which must be counted for two double points.

The converse of the theorem is evidently also true, i. e., *There always exists a quartic of the first kind in space, such that the rotations of its points about the vertical axis through certain definite angles proportional to the time, will lead to a given spherical pendulum curve, as specified above.*

UNIVERSITY OF ILLINOIS.

A THEOREM CONCERNING CONTINUOUS CURVES.

BY PROFESSOR ROBERT L. MOORE.

(Read before the American Mathematical Society, October 28, 1916.)

IN this paper I propose to show that every continuous curve has the simple property stated below in Theorem 1. Though my proof is worded for the case of a plane curve, it is clear that with a slight change in phraseology it would apply to a curve in a space of any number of dimensions.

LEMMA. *If $S_1, S_2, S_3, \cdots$ is a countable sequence of connected,* bounded point sets such that, for every n, S_n contains S'_{n+1},† then the set of all points that are common to $S_1, S_2, S_3, \cdots$ is closed and connected.*

For a proof of this lemma see my paper "On the foundations of plane analysis situs," *Transactions of the American Mathematical Society,* volume 17 (1916), page 137. Cf. also S. Janiszewski and E. Mazurkiewicz, *Comptes Rendus,* volume 151 (1910), pages 199 and 297 respectively.

THEOREM 1. *Every two points of a continuous curve are the extremities of at least one simple continuous arc that lies entirely on that curve.*

Proof. Suppose A and B are two points belonging to the continuous plane curve C. Hahn has shown‡ that the curve C is connected "im kleinen," i. e., that if P is a point of C, ϵ is a positive number and K is a circle, of radius $1/\epsilon$, with center at P, then there exists, within K and with center at P, another circle $K_{\epsilon P}$ such that if X is a point within $K_{\epsilon P}$, and belonging to C, then X and P lie together in some connected subset of C that lies entirely within K. Let $\overline{K}_{\epsilon P}$ denote the set of all points $[Y]$ belonging to C such that Y and P lie together in some connected subset of C that lies entirely within K. Clearly $\overline{K}_{\epsilon P}$ contains $K_{\epsilon P}$,

* A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

† If S is a point set, S' denotes the set of points composed of S together with all its limit points.

‡ Hans Hahn, "Ueber die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist," *Jahresbericht der Deutschen Mathematiker-Vereinigung,* vol. 23 (1914), pp. 318–322.

and indeed if Z is any point of $\overline{K}_{\epsilon P}$ then there exists about Z a circle such that every point of C within this circle belongs to $\overline{K}_{\epsilon P}$. Let $\epsilon = 1$ and for each point P of C construct the corresponding $\overline{K}_{1P}$. Let $\overline{G}_1$ denote the set of all such $\overline{K}_{1P}$'s. I will proceed to show that there exists a simple chain* from A to B every link of which is a point set of the set $\overline{G}_1$. Suppose that no such simple chain exists. Then the points of C fall into two classes, S_A and S_B where S_A is the set of all points $[P]$, belonging to C, such that P can be joined to A by a simple chain every link of which belongs to the set $\overline{G}_1$, while S_B is composed of all the remaining points of C. Since C is connected, one of the sets S_A and S_B contains a point X which is a limit point of the other one. There exists a circle k with center at X such that every point of C that lies within k lies also in $\overline{K}_{1X}$. But the interior of k contains a point A_1 belonging to S_A and a point B_1 belonging to S_B where A_1 is X or B_1 is X according as X belongs to S_A or to S_B. The point A can be joined to A_1 by a simple chain $R_1R_2R_3 \cdots R_n$ every link of which is a point set of the set $\overline{G}_1$. Let R_k be the first link of this chain that has a point in common with $\overline{K}_{1X}$. Then $R_1R_2R_3 \cdots R_k\overline{K}_{1P}$ is a simple chain from A to B_1 every link of which belongs to $\overline{G}_1$. Thus the supposition that A can not be joined to B by a simple chain every link of which is a point set belonging to $\overline{G}_1$ leads to a contradiction. It follows that A can be joined to B by at least one such chain $R_{11}R_{12} \cdots R_{1m_1}$. Call this chain C_1. For each i $(1 \leqq i < m_1)$ select a point P_{1i} common to R_{1i} and R_{1i+1} Let $P_{10} = A$ and $P_{1m_1} = B$. If $0 \leqq i < m_1$ then P_{1i} can be joined to P_{1i+1} by a simple chain C_{1i+1} each link of which is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq 2$ and lies with all its limit points entirely in R_{1i+1}. If any link of the chain C_{11}, except the last one, has a point in common with any link of C_{12}, then omit from C_{11} every link that follows T_{11}, where T_{11} is the first link of C_{11} that has a point in common with a link of C_{12}; also omit

* A *simple chain* from A to B is a finite sequence of point sets $R_1, R_2, R_3, \cdots R_n$ such that (1) R_1 contains A if and only if $i = 1$, (2) R_n contains B if and only if $i = n$, (3) if $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i < j$, then R_i has a point in common with R_j if and only if $j = i + 1$. The point set R_k $(1 \leqq k \leqq n)$ is said to be the kth link of the chain $R_1R_2R_3 \cdots R_n$ and the chain $R_1R_2R_3 \cdots R_n$ is said to join A to B. Cf. my paper "On the foundations of plane analysis situs," loc. cit., page 134.

from C_{12} every link (if there be any such) that precedes the last link that has a point in common with T_{11}. These omissions having been made, the remaining links of the chains C_{11} and C_{12} form a simple chain $\overline{C}_{12}$ from A to P_{12}. In a similar manner it may be shown that there exists a simple chain $\overline{C}_{13}$ from A to P_{13} such that each link of $\overline{C}_{13}$ is a link of either $\overline{C}_{12}$ or C_{13}. This process may be continued. It follows that there exists a simple chain C_2 from A to B such that each link of C_2 is a link of some C_{1i} $(1 \leqq i \leqq m_1)$. The chain C_2 has the property that each one of its links lies wholly in some single link of the chain C_1 and if a link x of C_2 lies in a link y of C_1 then every link that follows x in C_2 lies either in y or in some link that follows y in C_1. Similarly there exists a chain C_3 having a relation to C_2 analogous to the above indicated relation of C_2 to C_1 and such that every link of C_3 is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq 3$. This process may be continued. Thus there exists an infinite sequence of simple chains $C_1, C_2, C_3, \cdots$ such that (1) each link of the chain C_{n+1} lies, together with all its limit points, wholly in some single link of C_n; (2) if a link x of C_{n+1} lies in a link y of C_n then each link that follows x in C_{n+1} lies either in y or in some link that follows y in C_n; (3) every link of C_n is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq n$.

Let S_n denote the point set which is the sum of all the links of the chain C_n. Let S denote the set of all the points that the sets $S_1, S_2, S_3, \cdots$ have in common. It will be shown that S satisfies Lennes' definition* of a simple continuous arc from A to B.

I. That S is closed and connected follows easily with the help of the lemma on page 233. That it is bounded is evident.

II. To prove that S contains no connected proper subset that contains both A and B, let us first order the points of S. If X_1 and X_2 are two distinct points of S, then there exists n such that if $r \geqq n$ and P is a point of S then X_1 and X_2 do not both lie in $\overline{K}_{rP}$. But every link of C_n is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq n$. Hence for every two distinct points X_1 and X_2 be-

* A *simple continuous arc* from A to B is a bounded, closed, connected set of points containing A and B but containing no connected proper subset that contains both A and B. See N. J. Lennes, "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, vol. 33 (1911), p. 308, and this BULLETIN, vol. 12 (1906), p. 284.

longing to C there exists n such that X_1 and X_2 do not belong to the same link of C_n. Furthermore it is clear that if X_1 and X_2 do not lie in the same link of C_n but X_1 lies in a link of C_n that precedes one in which X_2 lies, then, if $m > n$, every link of C_m that contains X_1 precedes every link of C_m that contains X_2. The point X_1 is said to *precede* the point X_2 ($X_1 < X_2$) if there exists n such that every link of C_n that contains X_1 precedes every link of C_n that contains X_2. From facts observed above it follows that if X_1 and X_2 are distinct points of C then either $X_1 < X_2$ or $X_2 < X_1$; while if $X_1 < X_2$ then it is not true that $X_2 < X_1$. Furthermore if $X_1 < X_2$ and $X_2 < X_3$ then $X_1 < X_3$. For there exist n_1 and n_2 such that n_1 and n_2 do not lie in the same link of C_{n_1} and X_2 and X_3 do not lie in the same link of C_{n_2}. Hence every link of $C_{n_1+n_2}$ that contains X_1 precedes every link of $C_{n_1+n_2}$ that contains X_2 and every link of $C_{n_1+n_2}$ that contains X_2 precedes every one that contains X_3. Hence every one that contains X_1 precedes every one that contains X_3. Therefore $X_1 < X_3$.

Suppose now that H is a proper subset of S that contains both A and B. Then there exists a point P belonging to S, but different from A and from B, such that H is a subset of $S - P$. Now $S - P = S_A + S_B$ where S_A is the set of all points of S that precede P and S_B is the set of all points of S that follow P. It is clear that S_A contains A and S_B contains B. Suppose that P_A is a point of S_A. Then there exists n such that every link of C_n that contains P_A precedes every one that contains P. Suppose that some link y of the chain C_n contains P_A and also a point P_B of the set S_B. Since y precedes every link of C_n that contains P, it follows that P_B precedes P, which is contrary to hypothesis. Hence no link of C_n that contains P_A contains any point of S . But some link l of C_n does contain P_A. There exists, about P_A, a circle t such that every point of C that lies within t is a point of l. It follows that there is no point of S_B within the circle t. Therefore P_A is not a limit point of S_B. Similarly no point of S_B is a limit point of S_A. But H contains a point A that belongs to S_A and a point B that belongs to S_B. Moreover H is a subset of $S_A + S_B$. It follows that H is not connected.

It follows that S is a simple continuous arc from A to B. But clearly S is a subset of C. The truth of Theorem 1 is therefore established.

THE UNIVERSITY OF PENNSYLVANIA.

A NOTE ON DISCONTINUOUS SOLUTIONS IN THE CALCULUS OF VARIATIONS.

BY DR. PAUL R. RIDER.

(Read before the Southwestern Section of the American Mathematical Society, December 2, 1916.)

It is the object of this note to give the corner conditions and the forms of the Carathéodory Ω-function for Bliss's form of the simplest problem of the calculus of variations, and for an analogous form of the problem in space.

For the purpose of orientation and of introducing notation, a brief résumé of a part of the theory of discontinuous solutions as treated by Bolza* is given here.

1. If at a point $P_0(t_0)$ of a curve $x = x(t)$, $y = y(t)$ that minimizes or maximizes the definite integral

$$\int_{t_1}^{t_2} F[x(t), y(t), x'(t), y'(t)]dt$$

the curve possesses a corner, the corner conditions

$$F_{x'}\big|^{t_0-0} = F_{x'}\big|^{t_0+0}, \quad F_{y'}\big|^{t_0-0} = F_{y'}\big|^{t_0+0}$$

must be satisfied.

Let $P_1P_0P_2$ be an extremal (that is, a minimizing or maximizing curve) having a corner at P_0, the corner conditions being satisfied. Suppose that the continuity and other conditions usually imposed in the calculus of variations hold for the arcs P_1P_0, P_0P_2 and for the family of extremals

$$x = \phi(t, a), \quad y = \psi(t, a),$$

which contain the extremal arc $E_0 \equiv P_1P_0$ for $a = a_0$. Designate by τ_0 and $\bar{\tau}_0$ respectively the angles that the positive tangents to the arcs P_1P_0 and P_0P_2 at the point P_0 make with the positive x-axis. Then, if it is desired to find on E_a, a neighboring extremal to E_0, a point $P(t)$ and a direction $\bar{\tau}$ such that τ and τ (τ is the positive direction of E_a at P) satisfy the corner conditions, it is necessary to solve for t and $\bar{\tau}$ the equations

$$(1) \qquad F_{x'} - \bar{F}_{x'} = 0, \quad F_{y'} - \bar{F}_{y'} = 0,$$

* Bolza, Vorlesungen über Variationsrechnung, Chapter 8.

in which the arguments of $F_{x'}$, $F_{y'}$ are $\phi(t, a)$, $\psi(t, a)$, $\cos \tau(t, a)$, $\sin \tau(t, a)$, and of $\overline{F}_{x'}$, $\overline{F}_{y'}$ are $\phi(t, a)$, $\psi(t, a)$, $\cos \bar{\tau}$, $\sin \tau$. The functional determinant of (1) with respect to t, $\bar{\tau}$ at the point P_0 has the value

$$\sqrt{x'^2 + y'^2}\,\overline{F}_1(t_0)\Omega_0,$$

where $\Omega = \overline{F}_x \cos \tau + \overline{F}_y \sin \tau - F_x \cos \bar{\tau} - F_y \sin \tau$, and where the $\overline{F}_1$-function* has the arguments x_0, y_0, $\cos \bar{\tau}$, $\sin \bar{\tau}$. The subscript on the Ω-function denotes that the arguments ϕ, ψ, τ, $\bar{\tau}$, wherever they occur in it, have the values x_0, y_0, τ_0, $\bar{\tau}_0$ respectively. Since a usual assumption is that $F_1 \neq 0$ along an extremal, it follows that if $\Omega_0 \neq 0$ equations (1) can be solved uniquely for t and $\bar{\tau}$ and the solution $t = t(a)$, $\bar{\tau} = \bar{\tau}(a)$ will be continuous in the vicinity of $a = a_0$ and will satisfy the initial conditions $t(a_0) = t_0$, $\bar{\tau}(a_0) = \bar{\tau}_0$.

Thus can be obtained a broken extremal $E_a + \overline{E}_a \equiv P_1PP_2$ with a corner at the point P. If a is allowed to vary, a family of such extremals is obtained, and the point P describes a so-called corner curve whose parameter is a.

2. The extension of this theory to the case in which the integrand of the definite integral is a function of three variables and their derivatives is perfectly obvious and is readily accomplished, but it may be worth while to give it here.

For this case there are four equations

$$(2)\qquad \begin{gathered} F_{x'} - \overline{F}_{x'} = 0, \quad F_{y'} - \overline{F}_{y'} = 0, \quad F_{z'} - \overline{F}_{z'} = 0, \\ \cos^2 \bar{\alpha} + \cos^2 \bar{\beta} + \cos^2 \bar{\gamma} - 1 = 0, \end{gathered}$$

the arguments of $F_{x'}$, $F_{y'}$, $F_{z'}$ being $\phi(t, u, v)$, $\psi(t, u, v)$, $\chi(t, u, v)$, $\cos \alpha(t, u, v)$, $\cos \beta(t, u, v)$, $\cos \gamma(t, u, v)$, and those of $\overline{F}_{x'}$, $\overline{F}_{y'}$, $\overline{F}_{z'}$ being $\phi(t, u, v)$, $\psi(t, u, v)$, $\chi(t, u, v)$, $\cos \bar{\alpha}$, $\cos \bar{\beta}$, $\cos \bar{\gamma}$. At a point $P_0(t_0)$ the value of the functional determinant of the equations with respect to t, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ is found to be

$$2\sqrt{x'^2 + y'^2 + z'^2}\,\sin \alpha_0 \sin \beta_0 \sin \gamma_0\, \overline{F}_1(t_0)\Omega_0,$$

where

$$\Omega = \overline{F}_x \cos \alpha + \overline{F}_y \cos \beta + \overline{F}_z \cos \gamma - F_x \cos \bar{\alpha} - F_y \cos \bar{\beta} - F_z \cos \bar{\gamma}.$$

* For definition see Bolza, loc. cit., p. 196.

The $\overline{F}_1$-function* has the arguments x_0, y_0, z_0, $\cos \alpha_0$, $\cos \overline{\beta}_0$, $\cos \overline{\gamma}_0$. If the usual assumption that it is different from zero along an extremal is made, it follows that if $\sin \overline{\alpha}_0 \sin \overline{\beta}_0 \sin \overline{\gamma}_0 \neq 0$, that is, if the extremal arc P_0P_2 does not have an initial direction which is parallel to one of the three axes, and if $\Omega_0 \neq 0$, then the system (2) can be solved uniquely for t, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ in terms of u, v. A family of broken extremals can be determined as before, and the point P varies on a corner surface whose parameters are u, v.

3. For Bliss's form of the problem† the integral has the form

$$\int_{t_1}^{t_2} f(x, y, \tau)\sqrt{x'^2 + y'^2}dt,$$

where $\tau = \arctan y'/x'$. By the methods ordinarily employed the corner conditions that must be satisfied at a point P are found to be

$$f \cos \tau - f_\tau \sin \tau - \overline{f} \cos \tau + \overline{f}_\tau \sin \overline{\tau} = 0,$$
$$f \sin \tau + f_\tau \cos \tau - \overline{f} \sin \overline{\tau} - \overline{f}_\tau \cos \overline{\tau} = 0.$$

In these equations the parameter t, of the arguments of f, f_τ, $\overline{f}$, $\overline{f}_\tau$, has been replaced by s, the length of arc.

Their functional determinant with respect to s, $\overline{\tau}$ is

$$\overline{f}_1\omega,$$

where

$$f_1 = f + f_{\tau\tau}, \quad \omega = \overline{f}_x \cos \tau + \overline{f}_y \sin \tau - f_x \cos \overline{\tau} - f_y \sin \overline{\tau}.$$

Since here also $f_1 \neq 0$, the properties of ω are identical with those of the corresponding function in the Weierstrass form of the problem.

4. In a paper entitled "The space problem of the calculus of variations in terms of angle," to be published in the *American Journal of Mathematics*, I have considered an integral of the form

$$\int_{t_1}^{t_2} f(x, y, \tau, \sigma)\sqrt{x'^2 + y'^2 + z'^2}dt,$$

* For definition see Bliss and Mason, "The properties of curves in space which minimize a definite integral," *Transactions Amer. Math. Society*, vol. 9 (1908), p. 442.

† See Bliss, "A new form of the simplest problem of the calculus of variations," *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 405–414.

τ and σ being defined by the relations

$$\frac{x'}{\sqrt{x'^2+y'^2+z'^2}} = \cos\sigma\cos\tau,$$

$$\frac{y'}{\sqrt{x'^2+y'^2+z'^2}} = \cos\sigma\sin\tau,$$

$$\frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \sin\sigma.$$

Geometrically σ is defined as the angle that the positive tangent to a curve makes with its projection in the xy-plane, and τ is the angle that this projection makes with the positive x-axis. The corner conditions that must be satisfied by the extremals of such an integral are

$$f\cos\sigma\cos\tau - f_\tau\frac{\sin\tau}{\cos\sigma} - f_\sigma\sin\sigma\cos\tau$$
$$-\bar{f}\cos\bar{\sigma}\cos\bar{\tau} + \bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}} + \bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau} = 0,$$

$$f\cos\sigma\sin\tau + f_\tau\frac{\cos\tau}{\cos\sigma} - f_\sigma\sin\sigma\sin\tau$$
$$-\bar{f}\cos\bar{\sigma}\sin\bar{\tau} - \bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}} + \bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau} = 0,$$

$$f\sin\sigma + f_\sigma\cos\sigma - \bar{f}\sin\bar{\sigma} - \bar{f}_\sigma\cos\bar{\sigma} = 0.$$

Here again t has been replaced by s.

A somewhat lengthy piece of reckoning gives as the functional determinant of this system, with respect to s, $\bar{\tau}$, $\bar{\sigma}$,

$$-\frac{\bar{f}_1\omega}{\cos\bar{\sigma}},$$

where

$$f_1 = (f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma})$$
$$-(f_\tau\tan\sigma + f_{\tau\sigma})^2,$$

$$\omega = \bar{f}_x\cos\sigma\cos\tau + \bar{f}_y\cos\sigma\sin\tau + \bar{f}_z\sin\sigma$$
$$-f_x\cos\bar{\sigma}\cos\bar{\tau} - f_y\cos\bar{\sigma}\sin\bar{\tau} - f_z\sin\bar{\sigma}.$$

If we make the usual assumption $f_1 \neq 0$ (we assume also that $\cos\bar{\sigma} \neq 0$), the ω-function has the same properties as the corresponding function in three-dimensional space.

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THEORY OF THE TOP.

Advisory Committee for Aeronautics. Reports and Memoranda, No. 146. Report on Gyroscopic Theory. By SIR G. GREENHILL. Darling and Son, and T. Fisher Unwin, London, 1914. Folio. iii+277 pp.+49 figs. Boards. Price 10s.

The British Advisory Committee for Aeronautics has printed, or rather, until the war, had printed annual reports which are the prime sources for information regarding the development of the theory and practice of aeroplane design. Unfortunately they no longer divulge their findings. The Committee has also published other reports bearing less directly upon aeronautical problems, but yet advantageous in the more recondite parts of the theory. Of these supplementary publications, Greenhill's report on gyroscopic theory deserves particular attention of the student of mathematics and mechanics, and cannot forever be neglected by the practical engineer, so important are many instances of gyroscopic action becoming.

We are indeed fortunate that the Committee got Greenhill to prepare the report. His long-continued investigations on the top have made him a world-recognized authority on the subject. The fact that we have available for study the great work of Klein and Sommerfeld on the same subject does not in the slightest detract from the value of the present work. The merest glance will convince any reader that the two treatises are conceived in very different ways and that they rather supplement than overlap one another. Besides Greenhill's is decidedly shorter and more concerned with apparatus. It will be of more interest to the student of mechanics and engineering, though of less to the student of the theory of functions of a complex variable. And here we might remark that a beautiful theory and a practical analysis susceptible to ready calculation are unfortunately not always to be combined. The theta function is a thing of beauty, but I have always found the solution of the problem of the motion of the top by theta functions with complex arguments anything but satisfactory from the point of view of calculation. Perhaps after all Legendre knew what he was about when he introduced his third elliptic integral,—at any rate we judge that Greenhill thinks so.

There is a good deal of Greenhillian elliptic analysis in the report, and this may discourage a great many persons; but I desire particularly to emphasize the fact that there is also a great deal of the text which is practically free from elliptic integrals and functions, and which is almost arithmetic and algebraic in its simplicity and directness. The first chapter is on steady gyroscopic motion. Here we have 27 pages that anybody can read. This discussion abounds in the geometric and mechanical illustration for which the author is known. The distinction between strong and weak tops is brought out. The effect of the moving rotating parts of the Gnome engine upon the steering of an aeroplane is shown, also the similar phenomenon in the automobile. Other problems considered are: precession of the equinox, Serson-Fleurias gyroscopic horizon, how to roll a hoop, gyroscopic effect in wheeled carriages and in paddle or screw steamers, stability of an elongated body in a medium (airship or projectile or submarine boat). There is an appendix on general dynamical theory. Naturally, if one is to make this work simple he must concentrate his attention on the vector angular momentum; there is no other way, and the use of this vector is as easy as it is important. Greenhill's first chapter would be valuable for its emphasis on the angular momentum vector even if it contained nothing else of value.

The second chapter is on gyroscopic applications. The simple style of the first chapter is continued. We find treated: the Schlick sea gyroscope, the monorail carriage, the Bessemer saloon, with some mention of steering torpedoes. There is also a section on the theory of oscillation, free, damped, and forced, and an appendix on dynamical units. In both these chapters, as throughout the book, numerous references to the literature are given—to all sorts of literature, not merely to the mathematical.

Chapter III is on the general unsteady motion of the top. The integration rapidly leads to elliptic analysis. The author gives a summary of many ways of treating elliptic integrals and functions. By the mathematician this chapter may be added to the existing tables of formulas. There are, however, other points of interest: the discussion of nearly steady motion, Kirchoff's kinetic analogue of the twisted elastica, the whirling chain, and Clifford's vector treatment of the top. The fourth chapter is largely analytical geometry—Darboux's deformable

hyperboloid, Poinsot motion, curvature and inflexion of the herpolhode, curvature of the polhode, and the like. Chapters V and VI are respectively given to the algebraic cases of top motion, and to numerical illustrations and diagrams. This is the ellipsotomic problem. The cases discussed are: bisection of a period, trisection, quinquesection, seven, nine, and eleven section. Various improvements on the author's work in the *Annals of Mathematics* are given. The analysis will interest the mathematician, the figures will illustrate to anybody a number of cases of motion of the top. The applications are: nutation of the earth's axis, free Eulerian precession and variation of latitude, lunar nutation, and gyroscopic motion of the moon's node. In Chapter VII under the caption, the spherical pendulum, the author treats steady motion and small nutation, and the plane revolving catenary.

With the eighth chapter we come to motion referred to moving origin and axes, and the problems of the motion of a pendulum body and gyroscope about an axle on a whirling arm, of the sphere containing a flywheel and rolling on a table, of the modification of Poinsot motion when the body contains a flywheel spinning on an axle fixed in the body, of the expanding or contracting body, of Kelvin's liquid gyrostat, and finally of the motion of a solid, perforated or not, in an infinite liquid. A knowledge of elliptic functions is not particularly important for the reader of this chapter. In the following ninth and closing chapter Greenhill takes up dynamical problems of steady motion and small oscillations: general motion of a body rolling on a table with attention to steady motion and small oscillations, motion of a body sliding on a smooth table, the single track gyroscopic car, the gyrocompass, Gilbert's barogyroscope, the gyrostatic chain, Kelvin's flexible wire suspension, the gyroscopic pendulum with wire suspension, the stability of a revolving shaft, and the motion of a body with altazimuth suspension. In many of these problems elliptic functions are not mentioned.

It will be seen that the text divides itself naturally into two major parts. Chapters I, II, VIII, and IX deal with practical physical and engineering problems in a practical and on the whole in an elementary way without the use of elliptic functions. There are some sections of the other chapters, III to VII inclusive, which are also elementary, but as a whole these chapters are valuable chiefly to a reader who is not afraid of

detailed elliptic analysis. This twofold nature of the work is desirable, especially when the division is as clean cut as here, because each type of reader can find and use the part of the book he most needs and at the same time he can gain a glimpse, if not an appreciation, of the sort of work on the same problem which interests the other type of reader.

The forty-nine carefully executed figures are collocated upon large plates at the end of the book. For some of the illustrations this arrangement is satisfactory, but others of them should have been placed at the appropriate places in the text for the readers' ease in following demonstrations. The typography of the work seems excellent. We have now in English a great reference book on the top, a worthy companion and counterpart to that of Klein and Sommerfeld in German, one decidedly more compact and cheaper in price.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

NOTES.

THE second annual meeting of the Mathematical Association of America was held at Columbia University, Thursday to Saturday, December 28–30, 1916. Thursday afternoon was devoted to a joint session with the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science, and Thursday evening to a joint dinner of the four organizations. The programme of the Association included addresses by FLORIAN CAJORI: "Discussion of fluxions from Berkeley to Woodhouse"; M. W. HASKELL: "University courses in mathematics intended for teachers of secondary mathematics"; T. H. GRONWALL: "A nucleus for a mathematical library"; E. B. WILSON: "The mathematics of aerodynamics." On Friday there was an exhibition of portraits and medals of mathematicians from the collection of David Eugene Smith, the presentation and discussion of the report of the library committee, and a business meeting including the election of the following officers: President: FLORIAN CAJORI; vice-presidents: D. N. LEHMER and OSWALD VEBLEN; secretary-

treasurer: W. D. CAIRNS; members of the council: E. R. HEDRICK, E. V. HUNTINGTON, HELEN A. MERRILL, R. E. MORITZ, D. E. SMITH.

AT the meeting of the Edinburgh mathematical society on December 8 the following papers were read: By L. R. FORD: "On a class of continued fractions"; by E. T. WHITTAKER: "On certain determinants of Cayley and Sylvester"; by E. M. HORSBURGH: "An addition to the slide rule."

THE opening (January) number of volume 39 of the *American Journal of Mathematics* contains the following papers: "Linear difference and differential equations," by TOMLINSON FORT; "Some singularities of a contact transformation," by W. V. LOVITT; "Oscillations near an isosceles-triangle solution of the problem of three bodies as the finite masses become unequal," by DANIEL BUCHANAN; "The finite groups of birational transformations of a net of cubics," by L. C. COX; "On the determination of a certain class of surfaces," by A. E. YOUNG; "On orthoptic and isoptic loci," by HAROLD HILTON and R. E. COLOMB; "A new canonical form for systems of partial differential equations," by L. B. ROBINSON.

THE December number (series 2, volume 18, number 2) of the *Annals of Mathematics* contains the following papers: "The numerical factors of the arithmetic forms $\prod_{i=1}^{n} (1 \pm \alpha_i^m)$," by T. A. PIERCE; "A problem in geometry connected with the analytic continuation of a power series," by T. H. GRONWALL; "On the power series for $\log (1 + z)$," by T. H. GRONWALL; "On the convergence of Binet's factorial series for $\log \Gamma(s)$ and $\Psi(s)$," by T. H. GRONWALL; "Note on Cauchy's integral formula," by J. L. WALSH; "On the rational, integral invariants of nilpotent algebras," by O. C. HAZLETT; "On trigonometric series," by W. L. HART.

JOHN WILEY and Sons announce the publication of volumes on Elliptic Integrals by HARRIS HANCOCK and Differential and Integral Calculus by H. B. PHILLIPS.

PROFESSORS PAUL PAINLEVÉ, of the University of Paris, and VITO VOLTERRA, of the University of Rome, have been elected honorary members of the Royal Institution, of London.

PROFESSOR TOMLINSON FORT, of the University of Michigan, has been appointed head professor of mathematics in the University of Alabama.

MR. E. J. OGLESBY, of the University of Virginia, has been appointed professor of mathematics in William and Mary College.

DR. P. H. LINEHAN has been promoted to an assistant professorship of mathematics at the College of the City of New York.

MR. J. K. WHITTEMORE has been appointed instructor in mathematics in the Sheffield Scientific School.

MR. J. A. BULLARD and Mr. C. E. NORWOOD have been appointed instructors in mathematics in the U. S. Naval Academy.

CHARLES SMITH, master of Sidney Sussex College and author of well-known textbooks on algebra and analytic geometry, died November 13, at the age of seventy-two years.

PROFESSOR H. A. SAYRE, of the University of Alabama, died December 2. Professor Sayre had been a member of the American Mathematical Society since 1891.

CHARLES A. PITKIN, professor of mathematics and physics at Thayer Academy, South Braintree, Mass., since its opening in 1877, died November 5, aged sixty-three years.

BOOK CATALOGUES:—Henry Sotheran and Company, 140 Strand, London, catalogue 766, exact sciences, about 1800 titles.—John Wheldon and Company, 38 Great Queen Street, London, scientific transactions and periodicals, 800 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLZANO (B.). Werke herausgegeben von A. Höfler. I. Wissenschaftslehre in 4 Bänden. Band 1: Versuch einer ausfuhrlichen und grössten Theils neuen Darstellung der Logik mit steter Rücksicht auf deren bisherige Bearbeiter. Mit einer Vorrede des J. C. A. Heinroth. Erster Band, Sulzbach, 1837. Leipzig, F. Meiner, 1914. 8vo. 16+571 pp. Geb. M. 14.00

CAVALCANTI (A.). Mathématique ou logique. Essai d'un cours philosophique de calcul arithmétique d'après Auguste Comte. Traduction française par Auguste de Aranjo Gonçalves. Paris, Blanchard, 1916. 8vo. 25+827 pp. Fr. 14.00

COMTE (A.). See CAVALCANTI (A.).

DARBOUX (G.). See POINCARÉ (H.).

GONÇALVES (A. DE A.). See CAVALCANTI (A.).

HALSTED (G. B.). See LOBATSCHEVSKI (N.).

HEINROTH (J. C. A.). SEE BOLZANO (B.).

HÖFLER (A.). See BOLZANO (B.).

HOLBA (S.). Eine neue Bahn in das Reich der Algebra. Forschungen und Gedanken eines Unberufenen. Algebraische Lösung des Casus irreducibilis, der Gleichung 5. Grades und des Fermat'schen Problems. Budapest, Kommissionsverlag Ludwig Kókai, 1915. 64 pp. 8 vo. M. 2.50

KNOBLAUCH (J.). See WEIERSTRASS (K.).

KÖRNER (K.). Die Mathematik im Preussischen Lehrerbildungswesen. (Abhandlungen uber den mathematischen Unterricht in Deutschland veranlasst durch die internationale mathematische Unterrichtskommission, Band 5, Heft 7.) Leipzig, Teubner, 1916. 136 pp. M. 1.60

LEBON (E.). See POINCARÉ (H.).

LEWY (H.). Ueber die apriorischen Elemente der Erkenntnis. I: Die Stufen der reinen Anschauung. Erkenntnistheoretische Untersuchungen über den Raum und die geometrischen Gestalten. Leipzig, F. Meiner, 1914. 9+204 pp. 8vo. M. 6.00

LIETZMANN (W.). Die Ausbildung der Mathematiklehrer an den höheren Schulen Deutschlands. (Berichte und Mitteilungen veranlasst durch die internationale mathematische Unterrichtskommission. Erste Folge, XI.) Leipzig, Teubner, 1915. 14 pp.

LOBATSCHEVSKI (N.). Geometrical researches on the theory of parallels. Translated from the original by G. B. Halsted. New edition. Chicago, Open Court, 1914. $1.25

MACAULAY (F. S.). The algebraic theory of modular systems. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 19.) Cambridge, University Press, 1916. 4s. 6d.

MACMAHON (P. A.). Combinatory analysis. Volume 2. Cambridge, University Press, 1916. Royal 8vo. 340 pp. 18s.

NÖRLUND (N. E.). See POINCARÉ (H.).

POINCARÉ (H.). Oeuvres publiées par G. Darboux. Tome 2 publié avec la collaboration de N. E. Nörlund et de E. Lebon. Paris, Gauthier-Villars, 1916. 4to. 77+628 pp. Fr. 35.00

RIEBESELL (P.). Die mathematischen Grundlagen der Variations- und Vererbungslehre. (Mathematische Bibliothek, Nr. 24.) Leipzig, Teubner, 1916. 8vo. 45 pp. M. 0.80

ROTHE (R.). See WEIERSTRASS (K.).

WEIERSTRASS (K.). Mathematische Werke herausgegeben unter Mitwirkung einer von den königlichen Preussischen Akademie der Wissenschaften eingesetzten Kommission. Band 5: Vorlesungen über die Theorie der elliptischen Functionen. Bearbeitet von J. Knoblauch. Band 6: Vorlesungen über Anwendungen der elliptischen Functionen. Bearbeitet von R. Rothe. Mit einem Bildnis von Weierstrass. Berlin, Mayer und Müller, 1915. 8+327+9+354 pp. 4to.

II. ELEMENTARY MATHEMATICS.

BAUER (W.) und HANXLEDEN (E. v.). Oberstufe der Arithmetik. Braunschweig, Vieweg, 1913. 8vo. 8+208 pp. Geb. M. 3.60

——. Unterstufe der Arithmetik. Braunschweig, Vieweg, 1913. 8vo. 10+271 pp. Geb. M. 4.00

BRENKE (W. C.). See LONG (E.).

BÜTZBERGER (F.). Lehrbuch der ebenen Trigonometrie mit vielen Aufgaben und Anwendungen. Zürich, O. Füssli, 1916. 8vo. 98 pp. Fr. 2.00

——. Lehrbuch der Stereometrie. Zürich, O. Füssli, 1916. 8vo. 119 pp.

CRATHORNE (A. R.). See RIETZ (H. L.).

HANXLEDEN (E. v.). See BAUER (W.).

HEDSTRÖM (J. S.) och RENDAL (C.). Algebra för gymnasiet. Stockholm, A. Bonnier, 1915. 8vo. 245 pp.

JACKSON (L. L.). See YOUNG (J. W. A.).

LONG (E.) and BRENKE (W. C.). Plane geometry. (Correlated mathematics for secondary schools, Part 2.) New York, Century Company, 1916. 12mo. 8+276 pp. Cloth. $1.00

RENDAL (C.). See HEDSTRÖM (J. S.).

RIETZ (H. L.), CRATHORNE (A. R.) and TAYLOR (E. H.). School algebra. (American Mathematical Series.) Complete in one volume. New York, Holt, 1916. 8vo. 15+506 pp. $1.25

TAYLOR (E. H.). See RIETZ (H. L.).

TRIPARD (L.). Les mathématiques après l'école primaire. Paris, Dunod et Pinat, 1916. 12mo. 376 pp. Cartonné. Fr. 5.00

YOUNG (J. W. A.) and JACKSON (L. L.). A plane geometry. (Twentieth Century Text.) New York, Appleton, 1913.

III. APPLIED MATHEMATICS.

BEGGS (G. E.). Live load stresses in railway bridges. With formulas and tables. New York, Wiley, 1915. 5+123 pp. $2.00

BENOIT (C.) See LANCHESTER (F. W.).

BROWN (J. C.). See FINNEY (H. A.).

BRYAN (C. W.). See JOHNSON (J. B.).

CABRAL (A. F.). Cuadro sinoptico de la doctrina biocosmica de la gravitacion universal y de la jeneracion de los mundos. Santo Domingo, Imp. La Cuna de América, 1915. Folio. 17 pp.

CARY (E. R.). Geodetic surveying. New York, Wiley, 1916. 9+272 pp. Cloth. $2.50

CONKLIN (C. D.). Structural steel drafting and elementary design. New York, Wiley, 1915. 7+154 pp. Cloth. $2.50

FAWDRY (R. C.). Dynamics. Part 1. London, Bell, 1916. 8+177+9 pp. 3s.

FINNEY (H. A.) and BROWN (J. C.). Modern business arithmetic. Complete course. New York, Holt, 1916. 8vo. 6+488 pp. $1.10

GALÉ (J. G.). Matemáticas financieras. Primera parte: Intereses y anualidades ciertas. Buenos Aires, Libreria de Antonio Garcia Santos. Moreno y Bolivar, 1916. 226 pp.

GILTAY (J. W.). De strijkinstrumenten. Leyden, A. W. Lijthoff, 1916. 11+103 pp.

JOHNSON (J. B.), BRYAN (C. W.) and TURNEAURE (F. E.). The theory and practice of modern framed structures. Part 3: Design. 9th edition, rewritten by F. E. Turneaure and W. S. Kinne. New York, Wiley, 1916. 12+486 pp. Cloth. $4.00

KINNE (W. S.). See JOHNSON (J. B.).

KRUGER (L.). Transformation der Koordinaten bei der konformen Doppelprojektion des Erdellipsoids auf die Kugel und die Ebene. Leipzig, Teubner, 1914. 43 pp. 4to. M. 3.00

LANCHESTER (F. W.). Le vol aérien. Aérodonétique. Traduit de l'anglais sur la deuxième édition par C. Benoit. Paris, Gauthier-Villars, 1916. 8vo. 17+478 pp. Fr. 14.00

MELAN (J.) and STEINMAN (D. B.). Plane and reinforced concrete arches. New York, Wiley, 1915. 10+161 pp. Cloth. $2.00

MERRIMAN (M.). American civil engineers' pocket book. Edited by M. Merriman and sixteen associate editors. 3d edition, revised and enlarged. New York, Wiley, 1916. 1600 pp. Leather. $5.00

MOULTON (F. R.). An introduction to astronomy. Revised edition. New York, Macmillan, 1916. Cr. 8vo. 568 pp. $2.25

PHILIP (G.). Planisphere showing the principal stars visible for every hour in the year. London, G. Philip and Son, 1916. 1s. 6d.

ROHREBRG (A.). Theorie und Praxis des Rechenschiebers. (Mathematische Bibliothek, Nr. 23.) Leipzig, Teubner, 1916. 8vo. 50 pp. M. 0.80

ROWARTH (E.). Elements of engineering drawing. London, Methuen, 1916. 12+131 pp. 2s. 6d.

STEINMAN (D. B.). See MELAN (J.).

TURNEAURE (F. E.). See JOHNSON (J. B.).

WADDELL (J. A. C.). Bridge engineering. 2 volumes. New York, Wiley, 1916. 68+2177 pp. $10.00

WOLFF (G.). Mathematik und Malerei. (Mathematische Bibliothek, Nr. 20–21.) 1 volume. Leipzig, Teubner, 1916. 8vo. 76 pp.+4 plates. M. 1.60

THE DECEMBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE thirty-eighth regular meeting of the Chicago Section, being the seventh regular meeting of the Society at Chicago, took place at the University of Chicago on Friday, December 22. The meeting was attended by fifty-eight persons, a number only slightly below the usual total of attendance registrations. In view of the fact that the date of this meeting was very inconvenient, coming during the last days preceding Christmas, this nearly normal attendance may be looked upon as very encouraging. The following forty-two members of the Society were present:

Dr. Florence E. Allen, Professor F. Anderegg, Professor G. N. Bauer, Professor A. A. Bennett, Professor G. A. Bliss, Professor Henry Blumberg, Professor H. T. Burgess, Professor W. D. Cairns, Professor R. D. Carmichael, Mr. E. H. Clarke, Professor H. E. Cobb, Professor G. H. Cresse, Professor D. R. Curtiss, Dr. W. W. Denton, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Professor Arnold Emch, Professor W. B. Ford, Professor A. S. Hathaway, Professor T. H. Hildebrandt, Mr. Glenn James, Professor A. M. Kenyon, Dr. W. W. Küstermann, Professor Kurt Laves, Professor A. C. Lunn, Professor H. W. March, Professor William Marshall, Professor E. H. Moore, Professor C. C. Morris, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. A. Young.

The chairman of the Section, Professor W. B. Ford, presided at the two sessions. In the evening forty-seven members and friends dined together at the Quadrangle Club.

Plans are being made for including in the programme of the spring meeting a symposium on the subject of "The Lebesgue integral," to which an entire afternoon session is to be devoted.

The following papers were presented at this meeting:

(1) Dr. G. E. WAHLIN: "The cubic equation in a p-adic domain."

(2) Mr. GLENN JAMES: "Certain restrictions upon methods of summing divergent series."

(3) Dr. A. R. SCHWEITZER: "Some remarks concerning quasi-transitive functional equations."

(4) Professor H. T. BURGESS: "Note on the solution of the matrix equation $X^{-1}AX = N$."

(5) Professor HENRY BLUMBERG: "Certain general properties of functions."

(6) Professor A. S. HATHAWAY: "Certain residue groups."

(7) Professor E. B. VAN VLECK: "A proof of Haskins' momental theorem."

(8) Professor G. A. BLISS: "A device applicable to the problems of Lagrange and Mayer in the calculus of variations."

(9) Professor C. H. SISAM: "On the order of surfaces generated by systems of algebraic curves."

(10) Professor L. E. DICKSON: "On the history of the theory of numbers (preliminary communication)."

(11) Professor F. R. MOULTON: "An extended solution of differential equations."

(12) Professors G. N. BAUER and H. L. SLOBIN: "A system of algebraic and transcendental equations."

(13) Professor A. A. BENNETT: "Closed algebraic correspondences."

Dr. Wahlin's paper was read by title. Abstracts, numbered to correspond to the titles in the list above, follow below.

1. In this paper Dr. Wahlin makes a study of the general cubic equation for the domain of a rational prime p. He determines the conditions which the coefficients must satisfy in order that the cubic shall be irreducible or shall have one or three roots in $k(p)$. The result is then used and the necessary extension made for the complete determination of the factorization of p in the cubic domain.

2. Mr. James considers some of the necessary restrictions upon any method of summing series by the use of convergence factors, the assumption being made that the method satisfies "the conditions of consistency." In particular he proves that the limit of the sum of the convergence factors is unity, provided the method sums at least one convergent series, whose sum is not zero, to its ordinary sum and the limit of each of

the first K factors is zero. In the second part of his paper he shows that $\lim_{x=1} \Sigma a_n x^n$ cannot exist for a certain class of oscillating series.

3. Referring to a previous* paper, Dr. Schweitzer points out that it is possible to replace, in each theorem there given, all functional equations (in the hypothesis) but one by certain rational conditions on the M's. In particular, in the cubic case, the theorem as stated by the author (l. c.) remains valid if one of the functional equations is dropped from the hypothesis.

As subcategories of categories of functional equations previously† defined, the author defines (1) the quasi-transitive of degree $(n+1)(n=1,2,3,\cdots)$ and order $k, 0<k\leqq n+1$, (2) the transitive of degree $(n+1)$ and order $k, 0<k\leqq n+1$. The degree is defined as the number of arguments of any one of the functions involved in the equation; the order is the degree less the number of variables eliminated. If the order equals the degree, the concepts "quasi-transitive" and "transitive" coalesce and one has as class subordinate to the resulting class the "iterative and non-eliminative" equation. Examples: $\phi\{f(t, x_2, x_3), f(x_1, t, x_3), f(x_1, x_2, t)\} = f(x_1, x_2, x_3)$ is a transitive equation of degree three and order two; $\phi\{f(t_1, t_2, x_3), f(t_1, x_2, t_2), f(x_1, t_1, t_2)\} = f(x_1, x_2, x_3)$ is a transitive equation of degree three and order one; $f\{f(x_1, x_2, x_3), f(x_2, x_3, x_1), f(x_3, x_1, x_2)\} = f(x_1, x_2, x_3)$ is an iterative equation of degree three and order three.

The preceding classification of equations on the basis of order and degree necessarily involves the finiteness of the number of variables. It appears possible, however, to define at least formally extensions of categories of functional equations previously defined by the author to equations of infinite degree. In particular, one can give formal definitions of transitive and quasi-transitive equations of infinite degree and order k where k is a positive integer. For example, $\phi\{u_1, u_2, \cdots, u_n, \cdots\} = \psi(x_1, x_2, \cdots, x_n, \cdots)$ where $u_i = \lambda_i f(x_i, t_1, t_2, \cdots, t_n, \cdots)$ $(i = 1, 2, \cdots)$ is of infinite degree and order 1. The domain of validity of any of these formally defined equations is, of course, a matter of special investigation.

* Cf. BULLETIN, vol. 23 (1916), pp. 76–78.

† Cf. BULLETIN, previous abstracts.

On the basis of the properties of an abstract field, one defines directly an abstract relation having $n+1$ elements as arguments ($n = 1, 2, \cdots$) and whose functional interpretation is the equation $f\{u_1, u_2, \cdots, u_{n+1}\} = f(x_1, x_2, \cdots, x_{n+1})$, where $u_i = f\{t_1, t_2, \cdots, t_n, x_i\}$ ($i = 1, 2, \cdots, n+1$). In the latter relation $f(x_1, x_2, \cdots, x_{n+1}) = f_1\{u_{123\cdots n}, u_{2[n+1]}\}$, $n \geqq 2$, $u_{12} = f_1(x_1, x_2)$, $u_{123} = f_1(u_{12}, u_{23})$, $\cdots$, $u_{123\cdots n} = f_1(u_{123\cdots[n-1]}, u_{2n})$, and $f_1\{f_1(t, x_2), f_1(t, x_1)\} = f_1(x_1, x_2)$.

4. If A is any square matrix, there exists a similar matrix N which is of a well-known canonical form. In a paper in the *Annals of Mathematics*, September, 1916, Professor Burgess gave a practical method of determining the form of N for any given matrix A. In the present paper he gives a practical method for computing the matrix X such that $X^{-1}AX = N$.

The existence of X is well known, but no direct method computing it is to be found in the literature.

5. A résumé of Professor Blumberg's paper is contained in the *Proceedings of the National Academy of Sciences*, November, 1916. The paper will appear in full in the *Annals of Mathematics*.

6. Mr. Hathaway groups the residues of a prime $n = 6m + 2\epsilon + 3$ ($\epsilon = 1$ or 2), so that $x = a$, $1/a$, $-1-a$, $-1/(1+a)$, $-(1+a)/a$, $-a/(1+a)$, make a group. There are m groups of six residues, one of three residues $(1, -2, -\frac{1}{2})$, and ϵ of two residues, $(0, -1)$ and, when $\epsilon = 2$, $(\omega, \omega^2 = -1-\omega)$.

The function $y = (1+x+x^2)^3/(x+x^2)^2$ is invariant for a group, and designating the values for the $m+1$ groups by y_r, $r = 0, 1, \cdots, m$, in which $y_m = 27/4$, the invariant of the group of three, the numbers of quadratic and non-quadratic residues y_r are such that we may assign odd subscripts to the first, and even subscripts to the second.

Taking $\alpha = 3(m-r)+\epsilon$, $\beta = 2r+1$, $A_r = (2\alpha 3\beta)/n = \Gamma(\alpha+\beta)/\Gamma(\alpha+1)\Gamma(\beta+1)$, to define the functions, $\varphi(y) = \Sigma A_r y^{m-r}$, $F(y) = \Sigma\beta A_r y^{m-r}$, $F_1(y) = \Sigma\beta A_r y^{m-r}/(\beta+1)$, the congruences $F(y) \equiv 0$, $F_1(y) \equiv 0$, $8y^{m+1} - 3F_1(y) \equiv 0$, mod. n, are completely resolvable in y_r residues, the first in all except y_m, the second and third in quadratic and non-quadratic y's respectively.

If the sum of three nth powers be divisible by n^2, one of the powers, or the difference of the cubes of two of them, or the resultant of $\varphi(y) = 0$, $F(y) = 0$ is divisible by n.

7. Professor Van Vleck's paper gives a proof of the following theorem of Haskins (*Transactions*, April, 1916): Let $f(x)$ and $\phi(x)$ be two functions which are integrable in the sense of Lebesgue over the interval (a, b) and have upper and lower bounds,

$$h \leqq f(x) \leqq H, \qquad h \leqq \phi(x) \leqq H;$$

then if

$$\int_a^b [f(x)]^n dx = \int_a^b [\phi(x)]^n dx \qquad (n = 1, 2, 3, \cdots),$$

the measures of the sets of points $E_f(\alpha, \beta)$ and $E_\phi(\alpha, \beta)$ for which $f(x)$ and $\phi(x)$ lie between α and β inclusive will be equal for all pairs of values (α, β) included in the function range (h, H). Professor Van Vleck reduces this theorem to a special case of the problem of moments of Stieltjes and then establishes it by this author's results.

8. A class of curves in space of $n + 1$ dimensions may be defined by the conditions that each curve of the class must satisfy a set of differential equations of the form

$$\varphi_\alpha(x; y_1, \cdots, y_n; y_1', \cdots, y_n') = 0 \quad (\alpha = 1, 2, \cdots, m < n)$$

and pass through two fixed points. The problem of Lagrange is to characterize a curve of the class which makes an integral

$$\int f(x; y_1, \cdots, y_n; y_1', \cdots, y_n') dx$$

a maximum or minimum. It is customary to assume that along the particular curve in question the determinant of the first m rows of the matrix $\|\partial\varphi_\alpha/\partial y_k'\|$ $(\alpha = 1, \cdots, m; k = 1, \cdots, n)$ is everywhere different from zero. This is an unfortunately unsymmetric assumption, and it is the purpose of the paper of Professor Bliss to show that the details of the theory can be carried through if the matrix is only assumed to be of rank m at each point of the minimizing or maximizing curve. The device used is the addition of $n - m$ variables z_r $(r = m + 1, \cdots, n)$, and $n - m$ differential equations

$$\varphi_r(x; y_1, \cdots, y_n; y_1', \cdots, y_n') = z_r \quad (r = m + 1, \cdots, n),$$

such that the determinant $|\partial\varphi_i/\partial y_k'|$ $(i, k = 1, \cdots, n)$ is everywhere different from zero on the curve to be studied. The results are of course independent of the auxiliary equations and variables introduced. A similar method is applicable to the problem of Mayer.

9. Professor Sisam's paper discusses the relation between the order of the curves of a system of algebraic curves on an algebraic surface, the number of variable points common to two generic curves of the system, and the order of the surface. It determines a new sufficient condition that such a system of curves constitute a pencil.

11. In general, the proofs of the existence of the solutions of differential equations establish their existence only in limited domains. In this paper Professor Moulton gives a practical method, related closely to that known as Picard's method, for constructing the solutions so long as the independent and dependent variables remain in the region for which the differential equations have certain specified properties. By suitable specialization, the process reduces to the Cauchy-Lipschitz method. The validity of the method of mechanical quadratures employed by astronomers and physicists is established. Finally, corresponding results are established for a certain type of an infinite system of differential equations in infinitely many variables.

12. In the present paper Professors Bauer and Slobin study a system of three equations of the form

$$(A) \qquad \varphi_1(x, y) = 0, \quad \varphi_2(x, y) = 0, \quad T(x, y) = 0,$$

where the φ_i's are polynomials in x and y with algebraic coefficients, and where $T(x, y) = 0$ denotes an equation which, in general, is not satisfied if x and y are both algebraic numbers. As illustrations of equations of this type, may be mentioned

$$x - e^y = 0, \; y - \sin x = 0, \; \varphi_1(x, y) + \varphi_2(x, y) \tan \varphi_3(x, y) = 0.$$

$$\varphi_1(x, y)e^{\phi_2(x, y)} + \varphi_3(x, y) = 0.$$

The following are some of the theorems considered:

1. The system A cannot have simultaneous solutions (excepting possibly for the pairs of algebraic numbers, if any,

which satisfy $T(x, y) = 0$). No curves represented by equations of the form $\varphi_1(x, y) = 0$, $\varphi_2(x, y) = 0$ can intersect on the sine curve or the tangent curve excepting at the origin, nor on the exponential curve $x = e^y$ excepting in the point (1, 0).

2. The system $\varphi_1(x, y) + p(t)\varphi_2(x, y) = 0$, $\varphi_3(x, y) = 0$, $\varphi_4(x, y) = 0$ can have no simultaneous solutions except those, if any, which simultaneously satisfy $\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 0$, where $p(t)$ is a polynomial in t, a transcendental number.

3. All the singularities of the curves represented by $\psi(x, y) \equiv \varphi_1(x, y) + t\varphi_2(x, y) = 0$ which require $\partial\psi/\partial x = 0$ and $\partial\psi/\partial y = 0$, lie upon $\varphi_1(x, y) = 0$ and $\varphi_2(x, y) = 0$ if there are any at all.

4. All singularities of $\varphi_0(x, y) + p_1(t)\varphi_1(x, y) + \cdots + p_n(t)\varphi_n(x, y) = 0$ must lie upon each of the curves represented by the equations $\varphi = 0$, $\varphi_1 = 0$, $\cdots$, $\varphi_n = 0$ where $p_i(t)$, $(i = 1, \cdots, n)$, are polynomials in t, a transcendental number, the coefficients of the polynomials being algebraic numbers.

13. In this paper, Professor Bennett examines by elementary methods the form of a closed algebraic correspondence upon an algebraic curve or Riemann surface. As a result of certain elementary properties of involutions, and the group properties of closed algebraic and finite systems of points, the relations between closed correspondences and variable inscribed plane configurations are described. A systematic method of securing fundamentally different generalizations of the closure problems arising in connection with Poncelet polygons is obtained.

ARNOLD DRESDEN,
Secretary of the Section.

THE TWENTY-THIRD ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-third annual meeting of the Society, which was held in New York City on Wednesday and Thursday, December 27–28, 1916, was in several respects an exceptional occasion. It took place in the midst of the convocation week series of meetings of the American association for the advance-

ment of science and its long train of affiliated societies; and was immediately followed by the second annual meeting of the newly organized Mathematical association of America, with which the Society has not only a large common membership but also a general community of interest highly beneficial to both. The annual meeting is always one of the largest of the year, being the season of the election of officers and other members of the Council and the transaction of important business. This year it was especially marked by the delivery of the retiring address of President E. W. Brown, the subject of which was "The relations of mathematics to the natural sciences." This was presented before a joint session of the Society with the Mathematical association, the Astronomical society of America, and Section A of the American association and was followed by the retiring address of Vice-President A. O. Leuschner of Section A, on "Derivation of orbits—theory and practice." A joint dinner of the four organizations was held on Thursday evening at the Park Avenue Hotel, with an attendance of 143 members and friends. Much of the credit for the great success of the meetings is due to the joint committee on arrangements and to the programme committees of the Mathematical association.

Under all these favorable circumstances the attendance at the four sessions of the Society exceeded all previous records. It included the following 131 members and perhaps others whom the Secretary failed to identify in the multitude:

Professor R. B. Allen, Professor R. C. Archibald, Professor C. H. Ashton, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor R. D. Beetle, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. T. H. Brown, Professor Daniel Buchanan, Professor W. G. Bullard, Dr. R. W. Burgess, Professor W. D. Cairns, Professor Florian Cajori, Professor W. B. Carver, Professor A. B. Coble, Professor Abraham Cohen, Professor F. N. Cole, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Professor C. H. Currier, Professor T. W. Edmondson, Professor L. P. Eisenhart, Professor T. C. Esty, Professor G. C. Evans, Mr. G. W. Evans, Professor F. C. Ferry, Professor J. C. Fields, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor T. M. Focke, Professor W. B. Ford, Professor A. S. Gale, Professor W. A. Garrison, Professor O. E. Glenn, Mr. T. E. Gravatt,

Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. V. Gummere, Dr. W. L. Hart, Dr. Mary G. Haseman, Professor M. W. Haskell, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, President C. S. Howe, Professor L. A. Howland, Professor L. S. Hulburt, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. E. A. T. Kircher, Dr. J. R. Kline, Mr. E. H. Koch, Jr., Professor J. K. Lamond, Mr. Harry Langman, Professor Florence P. Lewis, Professor G. H. Ling, Dr. Joseph Lipka, Professor W. R. Longley, Dr. J. V. McKelvey, Professor Emilie N. Martin, Professor Helen A. Merrill, Dr. Mansfield Merriman, Professor E. J. Miles, Dr. A. L. Miller, Professor Bessie I. Miller, Professor G. A. Miller, Professor H. B. Mitchell, Professor H. H. Mitchell, Professor R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor Richard Morris, Mr. G. W. Mullins, Professor G. D. Olds, Professor F. W. Owens, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Professor J. M. Poor, Professor H. W. Reddick, Professor R. G. D. Richardson, Mr. J. F. Ritt, Professor E. D. Roe, Jr., Professor R. E. Root, Dr. J. T. Rorer, Professor D. A. Rothrock, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Mary E. Sinclair, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Professor Virgil Snyder, Professor Pauline Sperry, Dr. J. M. Stetson, Professor Henry Taber, Professor H. D. Thompson, Dr. J. I. Tracey, Professor C. B. Upton, Professor J. N. Van der Vries, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Dr. J. H. Weaver, Mr. H. E. Webb, Professor A. G. Webster, Dr. Mary E. Wells, Professor H. S. White, Professor E. E. Whitford, Mr. J. K. Whittemore, Professor A. H. Wilson, Professor E. B. Wilson, Professor Ruth G. Wood, President R. S. Woodward, Professor J. W. Young, Dr. Mabel M. Young.

President Brown occupied the chair, being relieved by Vice-Presidents Hedrick and Snyder and by Professor Olds. The Council announced the election of the following persons to membership in the Society: Professor H. H. Conwell, University of Idaho; Mr. Robert Dysart, Boston, Mass.; Dr. Mary G. Haseman, Johns Hopkins University; Mr. J. B.

Scarborough, North Carolina Agricultural and Mechanical College; Mr. J. J. Tanzola, U. S. Naval Academy. Ten applications for membership in the Society were received.

In response to an invitation received from the Department of Mathematics of the University of Chicago, it was decided to hold the summer meeting and colloquium of the Society at that university in 1919. A committee was appointed to arrange for the summer meeting of 1917. A committee was also appointed to proceed with the early publication of the Cambridge Colloquium Lectures; the volume will probably appear in the early summer.

The total membership of the Society is now 732, including 75 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 490; the number of papers read was 205. The number of members attending at least one meeting during the year was 278. At the annual meeting 235 votes were cast. The Treasurer's report shows a balance of $10,198.38, including the life membership fund of $6,039.87. Sales of the Society's publications during the year amounted to $1,434.28. The Library now contains 5,377 volumes, excluding unbound dissertations.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

President,	Professor L. E. DICKSON.
Vice-Presidents,	Professor A. B. COBLE, Professor E. B. WILSON.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1919,

Professor G. C. EVANS,
Professor G. H. LING,
Professor L. A. HOWLAND,
Professor R. L. MOORE.

The following papers were read at the annual meeting:

(1) Professor J. E. ROWE: "The relation of singularities of the rational quintic in space to loci of the rational plane quintic."

(2) Dr. C. A. FISCHER: "Linear functionals of n-spreads."

(3) Professor H. B. MITCHELL: "Geometrical limits for the imaginary roots of a polynomial with real coefficients."

(4) Professor ARNOLD EMCH: "A theorem on the curves described by a spherical pendulum."

(5) Mr. J. K. WHITTEMORE: "Spiral minimal surfaces."

(6) Dr. J. R. KLINE: "Concerning the complement of a countable infinities of point sets of a certain type."

(7) Professor L. L. DINES: "On projective transformations in function space."

(8) Professor C. C. GROVE: "Foundation of the correlation coefficient."

(9) Professor O. E. GLENN: "Preliminary report on invariant systems belonging to domains."

(10) Dr. NORBERT WIENER: "Certain formal invariances in Boolean algebras."

(11) Professor L. P. EISENHART: "Theory of transformations T of conjugate systems."

(12) Professor E. V. HUNTINGTON: "Complete existential theory of the postulates for serial order."

(13) Professor E. V. HUNTINGTON: "Complete existential theory of the postulates for well ordered sets."

(14) Professor DANIEL BUCHANAN: "Orbits asymptotic to an isosceles-triangle solution of the problem of three bodies."

(15) Professor DANIEL BUCHANAN: "Asymptotic satellites about the straight-line equilibrium points."

(16) Professor DANIEL BUCHANAN: "Asymptotic satellites about the equilateral-triangle equilibrium points."

(17) Dr. G. M. GREEN: "Isothermal nets on a curved surface."

(18) Dr. A. L. MILLER: "Systems of pencils of lines in ordinary space."

(19) Dr. W. L. HART: "On an infinite system of ordinary differential equations."

(20) Dr. W. L. HART: "Linear differential equations in infinitely many variables."

(21) Professor E. V. HUNTINGTON: "A set of independent postulates for cyclic order."

(22) Professor E. V. HUNTINGTON: "Sets of independent postulates for order on a closed line."

(23) Professor FRANK MORLEY: "The cubic seven-point and the Lüroth quartic."

(24) Professor J. L. COOLIDGE: "The intersections of a straight line and hyperquadric."

(25) Professor A. D. PITCHER: "Biextremal connected sets."

(26) Professor H. H. MITCHELL: "Proof that certain ideals in a cyclotomic realm are principal ideals."

(27) Professor H. H. MITCHELL: "On the asymptotic value of sums of power residues."

(28) Professor EDWARD KASNER: "Certain systems of curves connected with the theory of heat."

(29) Miss TERESA COHEN: "On a concomitant curve of the planar quartic."

(30) Professor P. F. SMITH: "A theorem for space analogous to Cesàro's theorem for plane isogonal systems."

(31) Professor W. E. STORY: "Some variable three-term scales of relation."

(32) Professor E. W. BROWN: presidential address: "The relations of mathematics to the natural sciences."

(33) Professor A. O. LEUSCHNER, vice-presidential address, Section A, American association for the advancement of science: "Derivation of orbits—theory and practice."

(34) Professor G. D. BIRKHOFF: "A class of series allied to Fourier series."

(35) Professor G. D. BIRKHOFF: "Note on linear difference equations."

(36) Professor G. A. MILLER: "Groups generated by two operators of the same prime order such that the conjugates of the one under the powers of the other are commutative."

(37) Mr. H. S. VANDIVER: "On the power characters of units in a cyclotomic field."

(38) Professor HENRY TABER: "On the structure of finite continuous groups."

Dr. Wiener was introduced by Professor Huntington, Miss Cohen by Professor Morley. Professor Leuschner's vice-presidential address was read by Professor Haskell. The papers of Professor Rowe, Professor Emch, Professor Dines, the first two papers of Professor Buchanan, Dr. Hart's first paper, and the papers of Professor Story and Professor Taber were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In a previous paper (*Transactions*, volume 13, No. 3, pages 395–404) Professor Rowe has pointed out how co-combinant curves of the rational plane quintic R_2^5, whose parametric equations are written *with* binomial coefficients, may be transformed into the parameters of singularities on the rational space quintic R_3^5, whose parametric equations are written *without* binomial coefficients. For instance, it was shown there that the point projection of a point of the plane upon the R_2^5 transforms into the parameters of the eight tetratactic planes of the R_3^5.

The present paper is a completion of part of the work suggested in the paper cited above. The simplest singularities of a rational plane curve are flex tangents, double tangents, and double points. The simplest singularities of a rational space curve are tetratactic planes, touching-osculating planes, tritangent planes, unisecant-tangent lines (i. e., lines tangent to the curve at one point and meeting it in one other point), and quadrisecant lines. Obviously the R_3^5 does not have any tritangent planes, but it has all of the other singularities. The author has shown how to derive the parameters of these from the proper covariant loci of the R_2^5.

The new results obtained may be stated briefly as follows: (1) The quartic (20), page 399 of the paper referred to above, transforms into a quartic whose roots are the parameters of the points cut out of the R_3^5 by its unique quadrisecant line. (2) Through every point of the plane pass twelve osculant conics of the R_2^5; the twelvic yielding the parameters of these transforms into the twelvic whose roots are the parameters of the twelve touching-osculating planes of the R_3^5. (3) Through every point of the plane pass twelve flex tangents of twelve osculant cubics of the R_2^5; the twelvic whose roots are the parameters of these cubics transforms into the twelvic whose roots are the parameters of the twelve unisecant-tangent lines of the R_3^5.

2. It has been proved by Riesz that a linear functional of a curve which is continuous with zeroth order can be expressed as a Stieltjes integral. The multiple Stieltjes integral has been defined by Fréchet, and proved to be a linear functional.

In the present paper Dr. Fischer has found some additional properties of such integrals, and proved that any linear functional of an n-spread which is continuous with zeroth order can be expressed as a multiple Stieltjes integral.

3. At the October meeting of the Society, Professor Mitchell presented a number of criteria whereby geometrical limits for the position of a pair of imaginary roots of an algebraic polynomial with real coefficients might be determined by inspection from the relative position of its real roots and bend points. In the present paper the method is extended so as to yield precise geometrical constructions when the polynomial has but two imaginary roots, and, in the case of more than two, to isolate the separate roots geometrically within regions that can be made small at will.

4. Professor Emch's paper appeared in full in the February BULLETIN.

5. It is well known that corresponding points of a family of associated minimal surfaces lie on an ellipse. In another paper Mr. Whittemore has found the locus of the vertices of these ellipses for any such family, and shown that, if the minimal surfaces are applicable to a surface of revolution, the locus consists of two parts, a plane and a surface of revolution. In the present paper it is proved that the only other minimal surfaces for which part of the locus is a plane are the spiral minimal surfaces, and that in this case the rest of the locus is a certain spiral surface. The author has then made a study of spiral minimal surfaces and proved certain properties of these surfaces, in many cases analogous to properties of minimal surfaces applicable to surfaces of revolution. These properties are related to the spiral curves of the surfaces, and concern lines of curvature, asymptotic lines, double lines, evolute surfaces, etc.

6. Dr. Kline's paper appears in full in the present number of the BULLETIN.

7. The transformations considered by Professor Dines are of the form

$$(1) \qquad f_1(x) = \frac{a(x) + b(x)f(x) + \int_0^1 c(x, y)f(y)dy}{d + \int_0^1 e(y)f(y)dy},$$

where the functions a, b, c, d, e satisfy the conditions:

1. All are continuous for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$;
2. The absolute value of $b(x)$ is greater than a positive constant on the interval $0 \leqq x \leqq 1$;
3. The equation $a(x) + b(x)f(x) + \int_0^1 c(x, y)f(y)dy = 0$ has a unique continuous solution $f(x) = h(x)$;
4. $d + \int_0^1 e(y)h(y) \neq 0$.

The set of all transformations of this type form a group under which the lines of function space are interchanged among themselves. Properties analogous to the properties of the corresponding projective transformations in space of n dimensions are obtained. Among other things it is shown that every finite transformation generated by the general infinitesimal projective transformation in function space, defined by Kowalewski, can be expressed in the form (1).

8. An analysis of the work of educational psychologists along the line of transfer and of the mathematics they have employed has led to the careful study of the foundations of the correlation coefficient, and cognate statistical measures discussed in this paper by Professor Grove.

9. Professor Glenn shows that, for a binary form, under general binary transformations, there exist complete systems of concomitants belonging to each of several domains of rationality. The methods by which the systems for these domains are constructed in case of the general transformation apply also to concomitants of substitutions constituting a special subset of the general transformations.

10. Dr. Wiener proves that the only operations in a Boolean algebra which possess formal properties such as logical addition and multiplication must possess are derivable from these by a one-one transformation belonging to the algebra. It is shown that negation remains invariant under all such transformations, and hence occupies a unique formal position in the algebra. The conditions under which an operation possesses such a unique formal position are discussed. The paper appeared in full in the January number of the *Transactions*.

11. When there is a one-to-one correspondence between the points of two surfaces S and S_1 of such a sort that the developables of the congruence of lines joining corresponding points meet S and S_1 in a conjugate system of curves on these surfaces, then S and S_1 are said to be in the relation of a transformation T. In several former papers Professor Eisenhart discussed certain types of these transformations, in particular those for which the conjugate systems on the two surfaces had their point invariants or their tangential invariants equal. In the present paper he develops the general theory of transformations T of a surface defined in either point or tangential coordinates. The well-known transformations D_m of isothermic surfaces are transformations T. The general transformations T admit a theorem of permutability reducing for the case of transformations D_m to the theorem due to Bianchi for the latter. The paper appeared in full in the January number of the *Transactions*.

12–13. Professor Huntington's two papers appear in full in the present number of the BULLETIN.

14. If two finite spheres of equal masses revolve about their common center of gravity, then an infinitesimal body subject to their attraction will describe a periodic orbit, under suitable initial conditions. This orbit is linear, being in the straight line through the center of gravity of the finite bodies and perpendicular to the plane of their motion. In a former paper (*Proceedings of the London Mathematical Society*, July, 1915), Professor Buchanan discussed the periodic oscillations about this linear orbit when the infinitesimal body is given an initial displacement from it. In the present paper, orbits are determined which are asymptotic to this linear orbit. The solutions are expansible as power series in a parameter which may denote the initial distance of the infinitesimal body from the linear orbit.

15. In this paper Professor Buchanan discusses orbits which are asymptotic to the periodic orbits of Class A and of Class B of the oscillating satellites about the straight-line equilibrium points as determined by Moulton in his Periodic Orbits, Chapter V.

16. In his third paper, Professor Buchanan discusses the

asymptotic satellites about the Lagrangian equilateral-triangle equilibrium points of the problem of three bodies. In Chapter IX of Moulton's Periodic Orbits, Buck determined two and three-dimensional periodic orbits about each equilibrium point. The two-dimensional periodic orbits were shown to exist only when one of the finite bodies is relatively small with respect to the other, but this restriction is not necessary for the existence of the three-dimensional orbits. In the present paper, however, it is shown that both the two and three-dimensional asymptotic orbits exist only when the finite bodies are more nearly equal than in the case of the two-dimensional periodic orbits. The two-dimensional asymptotic orbits approach the equilibrium points and not the corresponding periodic orbits. The three-dimensional asymptotic orbits, however, approach the three-dimensional periodic orbits.

17. An isothermal net on a curved surface is usually described as a net which divides the surface into infinitesimal squares. This of course is not properly speaking a geometric characterization of isothermal nets. Dr. Green gives two purely geometric characterizations of isothermal nets, one of which is easy to prove but requires a lengthy description; the other, however, though not so easy to prove, may be briefly described as follows. In previous papers Dr. Green has defined and made use of a certain geometric relation R, by means of which to a line l passing through any point of a curved surface is made to correspond a unique line l' in the corresponding tangent plane, the line l' being determined entirely by the line l and the parametric net of curves on the surface. Suppose the parametric net is any orthogonal net, and the line l is a normal to the surface. Then a unique line l' is determined in the corresponding tangent plane, so that corresponding to the congruence of normals l of the surface one obtains a congruence of lines l'. The developables of the congruence of lines l' correspond to a conjugate net on the surface, if and only if the parametric net is isothermal.

This obviously affords also an elegant characterization of isothermic surfaces, i. e., surfaces whose lines of curvature form an isothermal net. The other characterization of isothermal nets breaks down in this case; it also involves a use of the relation R, and the proof is so direct that the theorem

is interesting in spite of its failure when the lines of curvature are parametric.

18. The paper of Dr. Miller, in its three parts, treats of the projective differential properties of systems of ∞^1, ∞^2, and ∞^3 pencils of lines in ordinary space by means of the synthetic methods of Segre.

In part II the congruence determined by two nearby pencils of the system is found to be more interesting than, although closely allied to, the tangent linear congruence. The locus of the congruences determined by a pencil and all the infinitely near pencils of the system is found to be a tetrahedral complex of the type [(22) (11)]. Two interesting "developable" cases of these systems are discussed, one in which all the tangent linear congruences to the system at lines of a pencil have in common a pair of pencils with a common line, and one in which all these tangent congruences coincide. The locus of the centers of the pencils of the system and the envelope of their planes are found to be the focal surfaces of a director congruence to which all the pencils of the system are tangent in the first developable case, while in the second they coincide.

In part III the systems of ∞^3 pencils are classified according to the various kinds of two-parameter families of which they are the locus. This leads to a brief projective differential study of Pfaffian differential equations.

19. In a previous paper* Dr. Hart established the unique existence of a solution of the infinite system of ordinary differential equations

$$(1) \qquad \frac{dx_i}{dt} = f_i(x_1, x_2, \cdots; t) \qquad [x_i(t_0) = a_i;\ i = 1, 2, \cdots],$$

in which the variables were real and satisfied

$$(2) \qquad |t - t_0| \leqq b, \qquad |x_k - a_k| \leqq r_k < r.$$

In the former paper the results were obtained, by means of a generalization of the Picard approximation process, under the assumption that the f_i satisfied a continuity condition and, also, a condition with respect to the variables $(x_1, x_2, \cdots)$

* *Proc. National Academy of Sciences*, vol. 2 (1916), p. 309.

analogous to the classical Lipschitz condition. In the present paper there is established

Theorem I. Let the functions f_i be completely continuous in the region (2). Suppose that the maxima M_i (which exist) of the $|f_i(x_1, x_2, \cdots; t)|$ satisfy $M_i \leqq r_i M$. Then there exists *at least* one solution $\xi(t) = [x_1(t), x_2(t), \cdots]$ of (1), $x_i(t_0) = a_i$, which is defined and continuous for $t_0 \leqq t \leqq c$ where c is the smaller of b and $1/M$.

The solution is obtained by a generalization of the Cauchy polygon method; on adding the Lipschitz condition of the previous paper it is readily established that two solutions $\xi(t)$ cannot exist.

In the present paper Dr. Hart also treats a system of differential equations in general analysis. Among the particular instances of this general theory are found the case of a system of n differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n; t) \quad (i = 1, 2, \cdots, n),$$

where the f_i need not satisfy a Lipschitz condition, and also the case of system (1) under the hypotheses of Theorem I.

20. In considering finite systems of ordinary differential equations it is found that linear systems, in particular, have a great number of interesting properties. Many of these are connected with the notion of fundamental sets of solutions. Dr. Hart, in his second paper, considers a class of infinite systems of linear differential equations of the form

$$(1) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{\infty} k_{ij}(t)x_j + g_i(t) \quad (i = 1, 2, \cdots; |t| \leqq r),$$

in which $(x_1, x_2, \cdots)$ are complex values for which $\sum_{i=1}^{\infty} |x_i|^2$ converges. The infinite matrix $(k_{ij}(t))_{i,\, j=1,\, 2,\, \ldots}$ is assumed to be *limited* in the sense used for the term by Hilbert. The $(g_1(t), g_2(t), \cdots)$ are such that $\sum_{i=1}^{\infty} |g_i(t)|^2$ converges for every t. The elements k_{ij} and g_i are assumed to be analytic in t and are regular for $|t| \leqq r$. For the system (1) an infinite matrix of solutions $A(t)$ is said to constitute a fundamental set of solutions in case A is limited and possesses a unique limited

reciprocal. The system (1), in relation to this notion of fundamental sets of solutions, is found to have many of the familiar properties of finite systems of linear equations.

21. Professor Huntington shows that a class K is arranged in cyclic order (that is, in circular order with a definite sense around the circle) by a triadic relation $R(ABC)$, if the following five postulates are satisfied: (A) If ABC, then BCA. (B) If A, B, C are distinct, then at least one of the six triads, ABC, ACB, etc., is true. (C) If ABC is true, then ACB is false. (D) If ABC is true, then A, B, C are distinct. (1) If A, B, X, Y are distinct, and XAB and AYB, then XAY. The independence of these five postulates is established, and the relations between these postulates and the postulates for betweenness (BULLETIN, volume 23, pages 70–71, November, 1916) is discussed. A fuller abstract of the present paper may be found in the *Proceedings* of the National Academy of Sciences, volume 2, pages 630–631, November, 1916.

22. The theory of order on a closed line without distinction of sense is the same as the theory of the separation of pairs of points (begun by Vailati in 1895), and is based on a tetradic relation $ABCD$. Professor Huntington starts with the following basic list of fourteen postulates: (A_1) If $ABCD$, then $BCDA$. (A_2) If $ABCD$, then $DCBA$. (B) If A, B, C, D are distinct, then at least one of the twenty-four possible permutations $ABCD$, $ABDC$, etc., is a true tetrad. (C) If $ABCD$ is true, then $ABDC$ is false. (D) If $ABCD$ is true, then A, B, C, D are distinct. If A, B, C, X, Y are distinct, and $ABXC$ and $ABCY$, then: (1) $ABXY$; (2) $BXCY$; (3) $AXCY$. If A, B, C, X, Y are distinct, and $ABCX$ and $ABCY$, then: (4) $ABXY$ or $ABYX$; (5) $ACXY$ or $ACYX$; (6) $BCXY$ or $BCYX$; (7) $ABXY$ or $ACYX$; (8) $ABXY$ or $BCYX$; (9) $ACXY$ or $BCYX$. From this basic list, which contains many redundancies, eight different sets of independent postulates are obtained as follows (postulates A_1, A_2, B, C, D being required in each set): (I) 1, 4; (II) 1, 5; (III) 1, 6; (IV) 1, 7; (V) 1, 8; (VI) 1, 9; (VII) 2; (VIII) 3. Of these eight sets, the seventh appears to be the only one previously known.

23. Professor Morley's paper is in abstract as follows: Aronhold's construction of a curve of class four from seven

given points a_i gives a quartic of Lüroth's type when the seven points have each the same polar line as to a conic and a cubic (Bateman, *American Journal,* volume 36, 1914). If so, the nodal tangents of a cubic with a double point a_i and on the other six points will be apolar to the conic. Therefore a skew symmetric form $a^5x^2y^2$, which when y is a_i is the nodal tangents, will be such that the 6-rowed determinant of its coefficients vanishes. This determinant is the square of a Pfaffian of degree 15 in a_i, and further the Pfaffian vanishes when six of the points are on a conic. There is then a *cubic* invariant of 7 points which vanishes when they are an Aronhold set of a Lüroth quartic. This is the only cubic invariant of seven points. For six given points, there is then a cubic curve, on the six. Mapping on a cubic surface we have 36 plane sections (one for each double six), the locus of points on the surface from which the tangent cones are of Lüroth's type. Hence the degree of Lüroth's invariant is 54.

24. Professor Coolidge's paper contains a symmetrical form for the solution of $n - 1$ linear and one quadratic equation in $n + 1$ homogeneous variables.

25. In his thesis Fréchet considers the theory of functions μ of a variable q which ranges over a general class Q on which is defined a distance function δ conditioned to meet the needs of the theory. Professor Pitcher considers such systems $(Q; \delta)$ where δ is such that $\delta(qq) = 0$ and $\delta(q_1q_2) = \delta(q_2q_1)$. In particular he gives a set of three very simple conditions on the system $(Q; \delta)$ which are completely independent and which are both necessary and sufficient for a so-called uniformly proper theory of continuous functions on the set Q, where the term, uniformly proper, is used in a sense likely to be admitted by anyone who gives the matter careful consideration.

26. If l is a prime of the form $4n + 3$, and p a prime of the form $ml + 1$, Jacobi and Cauchy have shown that a certain power of a prime ideal factor of p in the quadratic realm $k(\sqrt{-l})$ is always a principal ideal. Dirichlet later showed that the exponent is the number of classes of ideals in the realm. The result first mentioned has been extended by Eisenstein and Stickelberger to primes not of the form $ml + 1$. Professor Mitchell obtains a more general result by

considering a cyclotomic realm of degree $2m$, determined by $2m$ periods, each containing n lth roots of unity, where n is odd, and $l = 2mn + 1$. If p is any prime that in this realm is the product of $2m$ conjugate prime ideal factors, he shows that a certain power of the product of m of these factors, if properly chosen, is a principal ideal. The exponent in this case is identical with the number given by Kummer as the so-called first factor of the class number.

27. If l is a prime of the form $4n + 3$, Dirichlet has shown that the sum of the quadratic residues of l that lie between 0 and l is always less than the sum of the non-residues in the same interval. Certain limits between which these sums must lie have been given by Stern, and somewhat similar limits may readily be deduced from some results obtained by Holden. Professor Mitchell obtains much closer limits for these sums, by means of which he shows that their ratio approaches the limit 1, as l increases indefinitely. Similar results are obtained for higher residues and a generalization is given for composite moduli.

28. The usual theory of isothermal systems of curves is based on the Laplace equation $\varphi_{xx} + \varphi_{yy} = 0$, derived on the assumption that the heat is in equilibrium. Professor Kasner studies more general systems of curves (for example the curves of constant temperature in a weather map) where this assumption is not valid. In particular, the cases where the rate of change of the temperature with respect to the time is independent either of the time or of the position are investigated in detail. In the first case we are led to a certain type of linear doubly infinite systems of curves. In the second case we have, as in the case of equilibrium, merely a simply infinite system $\varphi(x, y) =$ constant, but the fundamental partial differential equation is $\varphi_{xx} + \varphi_{yy} = 1$.

29. Miss Cohen's paper is in abstract as follows: The conic determined by a line ξ and the four tangents at its intersections with a quartic curve is of degree five in the coefficients of the quartic and eight in ξ. By use of special cases it can be found in terms of known comitants of the quartic. The conic breaks up when ξ is such that three of the tangents are on a point; therefore the discriminant of the conic is the locus of such

lines. The singular lines of this locus are: (1) the twenty-four stationary lines of the quartic, which are double lines with one contact, the flex; (2) the twenty-eight double lines of the quartic, which are double lines with the same contacts as with the quartic; (3) the twenty-one lines for which four tangents meet on a point, which are quadruple lines. The first two of these account for all the common lines of the locus and the quartic.

30. Cesàro's theorem for an isogonal system of plane curves is to the effect that the osculating circles of the curves through a given point pass through a second point. A companion theorem for an equitangential system has been established by Scheffers (*Mathematische Annalen*, volume 60 (1905), page 491), namely, the osculating circles of the curves of such a system which touch a given line touch also a second line. In Professor Smith's paper it is shown that simple transformations of the surface elements satisfying a partial differential equation of the first order lead, in one case, to an isogonal system of surface bands, in the other to an equitangential system of these bands. For each point of a given surface band two circles exist, each in a plane normal to the band at the point, such that the first circle osculates all surfaces containing the band, and the second osculates the developable surface upon which the band lies. Consideration of the circles of the first class for an isogonal system of surface bands leads to a theorem analogous to Cesàro's, while for an equitangential system of bands the circles of the second class play the rôle of the osculating circles in Scheffer's theorem.

31. Professor Story gives a proof of the theorem: an infinite succession of rational or irrational real numbers $y_1, y_2, y_3, y_4, \cdots$ of which each is expressible in terms of the preceding three by an equation of the form

$$y_h = L_h y_{h-1} + M_h y_{h-2} + N_h y_{h-3} \quad \text{for } 4 \leqq h$$

converges to a definite finite limit if the first three y's are finite and the coefficients L_h, M_h, N_h, for each integral suffix h as great as 4 are real numbers that satisfy the conditions

$$L_h + M_h + N_h = 1, \quad 0 < N_h < L_h, \quad 0 \leqq M_h, \quad \alpha < L_h,$$

where α is an a priori given positive proper fraction that is

appreciably greater than 0. The convergence of a regular continued fraction of the second order (in the cases in which such a fraction is usable) follows from the special case of this theorem in which $\alpha = \frac{1}{3}$.

32. President Brown's address appeared in full in the February BULLETIN.

34. The most notable properties of the series of orthogonal or biorthogonal functions defined by linear differential equations of the second order and linear boundary conditions are: (1) the functions satisfy these boundary conditions, (2) they have a certain asymptotic form, and (3) the series may be used to represent wide classes of arbitrary functions.

The functions appearing in the series are also given as the solutions of a homogeneous linear integral equation of Fredholm type with the corresponding Green's function $G(x, y)$ for kernel.

Professor Birkhoff shows that, if $H(x, y)$ be any continuous kernel satisfying the given boundary conditions and possessing a discontinuous first derivative for $x = y$ like the Green's function, the solutions of the integral equation in H will form an orthogonal or biorthogonal series with properties (1), (2), (3), at least if simple further conditions be imposed. A converse statement is also established.

35. In the theory of linear difference equations two fundamental sets of solutions have been defined by means of their asymptotic properties in a half plane bounded by a straight line parallel to the axis of imaginaries and lying in the complex plane of the independent variables. Professor Birkhoff shows that similar sets of solutions exist for half planes bounded by any line not parallel to the axis of reals, and that these solutions may be made to take the place of the fundamental sets referred to.

36. Professor Miller's paper appears in full in the present number of the BULLETIN.

37. If p is a prime integer, $\alpha = e^{2i\pi/p}$, and $\mathfrak{p}$ is an ideal prime in the algebraic field Ω defined by α, then

$$\omega^{[N(\mathfrak{p})-1]/p} \equiv \alpha^a \pmod{\mathfrak{p}},$$

where a is some integer less than p, ω is an integer in Ω, and $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$. Then we write

$$\alpha^a = \{\omega/\mathfrak{p}\}.$$

If Ω is a *regular* field and ω is a unit therein, then an explicit expression was given by Kummer for the exponent a. Let

$$\{\omega/\mathfrak{m}\} = \{\omega/\mathfrak{p}\}\{\omega/\mathfrak{q}\}\{\omega/\mathfrak{r}\}\cdots \quad (\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{r}\cdots),$$

where $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\cdots$, are prime ideals in any field Ω. Mr. Vandiver obtains in the present paper an explicit expression for $\{\omega/\mathfrak{m}\}$, where $\mathfrak{m}$ is a principal ideal in *any* cyclotomic field defined by a pth root of unity, and ω is included among certain types of units in the field.

38. Let $X_1, \cdots, X_r$ be the infinitesimal transformations of a finite continuous group with no exceptional infinitesimal transformation; and let $A_1, \cdots A_r$ be the matrices of any system whatever of independent infinitesimal linear homogeneous transformations generating a group with the same structure as G. The matrices $E_1, \cdots, E_r$ of the infinitesimal transformations of the group Γ adjoint to G constitute such a system. Denoting by SM the sum of the constituents in the principal diagonal of any matrix M, Professor Taber finds that $S(\Sigma\alpha_i A_i)^m$, for any positive integer m, is an invariant of the adjoint group Γ, and thus the coefficients of the characteristic equation of the general infinitesimal transformation $\Sigma\alpha_i A_i$ of the group $A_1, \cdots, A_r$ are invariants of Γ. Among these invariants one is of especial importance, namely, the quadratic form $S(\Sigma\alpha_i A_i)^2$. If the discriminant of this form is zero, and of nullity p, there are just p independent solutions of the equations $S(\Sigma\alpha_i A_i)A_k = 0$ $(k = 1, 2, \cdots, r)$; and if $\Sigma\alpha_i^{(q)}A_i$ for $q = 1, 2, \cdots, p$, are any p independent solutions of these equations, then $\Sigma\alpha_i^{(q)}X_i$, for $q = 1, 2, \cdots, p$, generate an invariant subgroup of G with p parameters. On the other hand, if the discriminant is not zero, the group G is perfect; and is either simple or semi-simple, unless possibly G contains an invariant subgroup every infinitesimal transformation of which is exceptional. For the case in which $A_1, \cdots, A_r$ are respectively identical with $E_1, \cdots, E_r$, these theorems are known. When the discriminant is not zero, the adjoint group is orthogonal, or becomes so by a suitable choice of the infinitesimal transformations of G; and in this case, if c_{ijk}

for $i, j, k = 1, 2, \cdots, r$, are the structural constants of G, we have $c_{ijk} + c_{ikj} = 0$.

F. N. COLE,
Secretary.

COMPLETE EXISTENTIAL THEORY OF THE POSTULATES FOR SERIAL ORDER.

BY PROFESSOR EDWARD V. HUNTINGTON.

(Read before the American Mathematical Society, December 27, 1916.)

THE purpose of this note is to establish the "complete independence"—in the sense defined by E. H. Moore*—of each of three different sets of postulates for serial order.

The first set of postulates (set A) is new and will be found more convenient for many purposes than either of the other sets. Set B dates back to Vailati, 1892.† Set C (a modification of set B and now widely used) was introduced by the present writer in 1905.‡

The universe of discourse considered in each of these sets is the universe of all systems (K, R), in which K is a class of elements, $A, B, C, \cdots$, and R is a dyadic relation; the notation $R(AB)$, or briefly AB, meaning that the relation R holds

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* E. H. Moore, "Introduction to a form of general analysis," Yale University Press (1910), p. 82. An interesting example of a proof of complete independence is given by R. D. Beetle, "On the complete independence of Schimmack's postulates for the arithmetic mean," *Math. Annalen*, vol. 76 (1915), pp. 444–446. [Compare R. Schimmack, "Der Satz vom arithmetischen Mittel in axiomatischer Begründung," *Math. Annalen*, vol. 68 (1909), pp. 125–132.] For a similar discussion of an *almost* completely independent set of postulates, see L. L. Dines, "Complete existential theory of Sheffer's postulates for Boolean algebras," this BULLETIN, vol. 21 (1915), pp. 183–188. [Compare H. M. Sheffer, "A set of five independent postulates for Boolean algebras, with application to logical constants," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 481–488.]

† G. Vailati, "Sui principî fondamentali della geometria della retta," *Rivista di Matematica*, vol. 2 (1892), pp. 71–75; B. Russell, Principles of Mathematics, vol. 1 (1903), pp. 203, 218–219.

‡ E. V. Huntington, "The continuum as a type of order," reprinted from the *Annals of Mathematics*, vols. 6 and 7 (1905), especially vol. 6, pp. 157–158; second edition, Harvard University Press, 1917, pp. 10–11, J. W. Young, Fundamental Concepts of Algebra and Geometry (1911), p. 68; A. N. Whitehead and B. Russell, Principia Mathematica, vol. 2 (1912), p. 513. (In the present terminology of Whitehead and Russell, a relation which satisfies postulate 1 is said to be "contained in diversity.").

for the elements A and B, in the order stated. The "postulates for serial order" are the conditions which such a system (K, R) must satisfy in order to be called a "serial system." If these conditions are satisfied, the relation R is called a serial relation with respect to the class K, and the notation $R(AB)$, or AB, may then be replaced by the more familiar notation $A \prec B$ ("A precedes B").

SET A. (POSTULATES 1, 2, 3, 4.)

The first postulate in this set concerns only a single element; the next two concern two distinct elements; and the fourth concerns three distinct elements.

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

That is, if A is an element of the class K, then the statement AA is always false.

Postulate 2. $A \neq B : \supset : AB \vee BA.$ (Connexity.)

That is, if A and B are distinct elements of K, then at least one of the statements AB and BA will be true.

Postulate 3. $A \neq B \,.\, AB \,.\, BA \,.=.\, 0.$

(Asymmetry for distinct elements.)

That is, if A and B are distinct elements of K, then not both the statements AB and BA can be true.

Postulate 4. $A \neq B \,.\, A \neq C \,.\, B \neq C \,.\, AB \,.\, BC : \supset : AC.$

(Transitivity for distinct elements.)

That is, if A, B, and C are distinct elements of K, then AB and BC together imply AC.

SET B. (POSTULATES 2, 3*a*, 4*a*.)

This set contains only three postulates, 3*a* and 4*a* being more inclusive forms of 3 and 4.

Postulate 2. $A \neq B : \supset : AB \vee BA.$ (Connexity.)

Postulate 3*a*. $AB \,.\, BA : = : 0.$

(Asymmetry for all elements.)

That is, if A and B are any elements of K (whether distinct or not), then not both the statements AB and BA can be true.

Postulate 4*a*. $AB \,.\, BC : \supset : AC.$

(Transitivity for all elements.)

That is, if A and B are any elements of K (whether distinct or not), then AB and BC together imply AC.

Here $3a$ implies 3 and 1, and $4a$ implies 4; also 1 and 3 imply $3a$, and 4 and 3 imply $4a$; so that sets A and B are clearly equivalent.

Note also that in set B, $4a$ may be replaced by 4, giving another set, B', comprising postulates 2, $3a$, and 4.

Set C. (Postulates 1, 2, $4a$.)

This set is a little simpler than set B, since postulate 1 is a little simpler than postulate $3a$; but neither set B nor set C is as explicit as set A.

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

Postulate 2. $A \neq B : \supset : AB \smile BA.$ (Connexity.)

Postulate $4a$. $AB \,.\, BC : \supset : AC.$

(Transitivity for all elements.)

Here 1 and $4a$ imply $3a$; and $3a$ implies 1; so that sets B and C are equivalent.

The question of the "complete independence" of each of these sets is in reality a question of classification. Every system (K, R), in the universe considered, either satisfies or fails to satisfy each of the postulates of each set. Now set A contains four postulates, and sets B and C each contain three. Hence *set A divides the universe theoretically into 16 compartments, while each of the sets B and C divides it into 8 compartments.* The purpose of this note is to show that no one of these compartments is "empty"; that is, to show that all the types of system (K, R) which can be distinguished by means of the postulates of any one of these three sets are actually represented among existing systems.

To prove this for set A, we exhibit 16 examples of systems (K, R); the first 8 of these answer also for sets B and C. In each of these examples, the class K is supposed to consist of only three elements, marked 1, 2, 3; the relation R is defined in each case by tabulating all the dyadic statements that are supposed to be true in that case. The character of each example is then shown in the table below, in which a dot (.) indicates that a postulate holds, and a cross (X) that it fails.

An examination of this table shows that examples of all the required types exist. (The entries below the double line are not necessary for the proof.)

In each of the examples shown, the class K consists of only

TABLE I.

Example.		Character.		
No	Description of R. ($K = 1, 2, 3$)	Set A.	Set B.	Set C
		1 2 3 4	2 3a 4a	1 2 4a
1	12, 13, 23.	. . . .	. . .	. . .
2	11, 22, 33, 12, 13, 23.	X . . .	. X .	X . .
3	12.	. X . .	X . .	. X .
4	11, 22, 33, 12.	X X . .	X X .	X X .
5	12, 31, 23.	. . . X	. . X	. . X
6	11, 22, 33, 12, 31, 23	X . . X	. X X	X . X
7	12, 23.	. X . X	X . X	. X X
8	11, 22, 33, 12, 23.	X X . X	X X X	X X X
9	12, 13, 23, 21, 31, 32.	. . X .	. X X	. . X
10	11, 22, 33, 12, 13, 23, 21, 31, 32.	X . X .	. X .	X . .
11	12, 21.	. X X .	X X X	. X X
12	11, 22, 33, 12, 21.	X X X .	X X .	X X .
13	12, 31, 23, 21.	. . X X	. X X	. . X
14	11, 22, 33, 12, 31, 23, 21.	X . X X	. X X	X . X
15	12, 13, 21, 31.	. X X X	X X X	. X X
16	11, 22, 33, 12, 13, 21, 31.	X X X X	X X X	X X X

three elements. It is easy to construct similar examples, however, in which K contains any number of elements, marked $1, 2, 3, \cdots n$, where n may be either finite ($n > 3$), or denumerably infinite. This may be done by adding the following items to the "descriptions of R" in the examples given above: in the examples which satisfy postulate 2, add xy whenever either x or y is > 3 and $x < y$; in examples 9 and 10 add also xy whenever either x or y is > 3 and $x > y$; in each of the even-numbered examples, add xx whenever $x > 3$. The character of these modified examples remains unchanged.

If we desire a *categorical* set of postulates for a *finite* series, we have merely to add to any one of our sets of postulates a postulate demanding that the system (K, R) shall contain exactly n elements ($n > 3$). The modified examples just

described show that each such categorical set of postulates will be also "completely independent."

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COMPLETE EXISTENTIAL THEORY OF THE POSTULATES FOR WELL ORDERED SETS.

BY PROFESSOR EDWARD V. HUNTINGTON.

(Read before the American Mathematical Society, December 27, 1916.)

A SYSTEM (K, R), where K is a class of elements $A, B, C, \cdots$ and R is a dyadic relation, is called a *well ordered* system when the following conditions are satisfied:*

(*a*) *the system* (K, R) *is a series; and*

(*b*) *every subsystem of* (K, R) *has a leading element.*†

Now when condition (*b*) is added to the conditions (*a*) which define a series, some of the conditions (*a*) become redundant. After eliminating these redundancies, we find the following *three sets of independent postulates for well ordered systems,* each of these three sets being in fact "completely independent" in the sense of E. H. Moore. (The numbering of the postulates is made to conform with that in the preceding note.)

SET I. (POSTULATES 1, 3, 5.)

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

Postulate 3. $A \neq B \,.\, AB \,.\, BA :=: 0.$ (Asymmetry for distinct elements.)

Postulate 5. *Every subsystem has at least one leading element.* ("Leadership," or the property of being "supplied with leaders.")

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* G. Cantor, *Math. Annalen*, vol. 49 (1897), p. 208. A. N. Whitehead and B. Russell, Principia Mathematica, vol. 3 (1913), p. 4.

† Here by a *series* we understand any system (K, R) which satisfies any one of the sets of postulates mentioned in the preceding note. A *subsystem* of (K, R) means any system (K', R') such that K' is a subclass of K, and $R' = R$. (Here K' is called a *subclass* of K if every element of K' belongs to K; that is, a subclass is either a part or the whole.) A *leading element* of a system means any element X having the following property: whenever Y is any other element of the system, then $R(XY)$, or simply XY, will be true. (If a system contains only a single element X, then X is a leading element of that system.)

SET II. (POSTULATES 3*a*, 5.)

Postulate 3*a*. $AB \,.\, BA :=: 0.$
(Asymmetry for all elements.)

Postulate 5. *Every subsystem has at least one leading element.*
("Leadership.")

Here 3*a* implies 1 and 3; and 1 and 3 together imply 3*a*. Hence set II is equivalent to set I.

SET III. (POSTULATES 1, 5*a*.)

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

Postulate 5*a*. *Every subsystem has just one leading element.*
("Unique leadership," or the property of being "supplied with unique leaders.")

Here 5*a* implies 3; for, if AB and BA were both true, and $A \neq B$, the subsystem consisting of the elements A and B would have two leading elements, contrary to 5*a*. Moreover, 3 and 5 imply 5*a*; for, if any subsystem had more than one leading element, say X_1 and X_2, then we should have X_1X_2 and X_2X_1, contrary to 3. Hence, set III is equivalent to set I.

It remains to show that every system that satisfies postulates 1, 3, and 5 will satisfy also the missing postulates for serial order, namely 2 and 4:

Postulate 2. $A \neq B :\supset: AB \vee BA.$ (Connexity.)

Postulate 4. $A \neq B \,.\, A \neq C \,.\, B \neq C \,.\, AB \,.\, BC :\supset: AC.$
(Transitivity for distinct elements.)

Here 2 follows from 5. For, if neither AB nor BA were true, and $A \neq B$, then the subsystem composed of A and B would have no leading element.

Also, 4 follows from 3 and 5. For, if AC were false, then CA would be true, by 5; but from the truth of AB, BC, and CA, would follow the falsity of BA, CB, and AC, by 3, and hence the subsystem A, B, C would have no leading element.

Thus we see that any one of the sets I, II, III is equivalent to the usual requirement represented by (*a*) and (*b*).

Finally, each of the sets I, II, III is "completely independent" in the Moorean sense, as is shown by the following examples, selected from the list used for another purpose in the preceding note.

TABLE II.

Example (K, R).		Character.		
No.	Description of R. ($K=1, 2, 3.$)	Set I.	Set II.	Set III.
		1 3 5	3a 5	1 5a
1	12, 13, 23.	. . .	. .	. .
2	11, 22, 33, 12, 13, 23.	X . .	X .	X .
5	12, 31, 23.	. . X	. X	. X
6	11, 22, 33, 12, 31, 23.	X . X	X X	X X
9	12, 13, 23, 21, 31, 32.	. X .	X .	. X
10	11, 22, 33, 12, 13, 23, 21, 31, 32.	X X .	X .	X X
11	12, 21.	. X X	X X	. X
12	11, 22, 33, 12, 21.	X X X	X X	X X

An inspection of this table shows that all the types of systems required by Moore's "complete existential theory" for each of the three sets of postulates actually exist. (The entries below the double line in the table are not necessary for the proof.)

In conclusion we note that, by the same device as that used in the preceding note, each of these examples may be enlarged so as to contain n elements ($n > 3$, finite or denumerably infinite), without altering the character of the example. Hence we may readily obtain "categorical" sets of completely independent postulates for every *finite* well ordered system.

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GROUPS GENERATED BY TWO OPERATORS OF THE SAME PRIME ORDER SUCH THAT THE CONJUGATES OF ONE UNDER THE POWERS OF THE OTHER ARE COMMUTATIVE.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, December 28, 1916.)

§ 1. *Introduction.*

In a previous article,* devoted to a study of conjugate operators, attention was called to the fact that a complete set of conjugate operators which are not all relatively commutative is transformed under the group according to a substitution group which is at most *triply* transitive. It is easy to prove that this substitution group is at most *doubly* transitive, and, as the p conjugates of order 2 under the holomorph of the group of the prime order p are transformed under this holomorph according to a doubly transitive group, the substitution group in question can actually be doubly transitive.

The most general definition of a complete set of conjugate operators, or conjugate subgroups, of a group G is that the set is composed of the totality of operators, or subgroups, which correspond under one or more of the possible automorphisms of G. It is, however, usually assumed that such a set is composed of the totality of operators, or subgroups, which correspond under at least one of the *inner* automorphisms of G. Since every possible automorphism of any group is an inner automorphism under its holomorph, this difference in the possible definitions of the expression *complete set of conjugates* does not affect the general theorems relating to such sets.

The theorem mentioned in the first paragraph can be materially extended by the consideration of a complete set of conjugate operators involving an operator which is commutative with at least one of the others without being commutative with all of them. When this condition is satisfied the group of transformations of the set is always *simply* transitive. To prove this let s_1, s_2 represent two commutative operators of

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 19 (1910), p. 318.

the set of conjugates under consideration, while s_2, s_3 are two non-commutative operators of this set; then the group G containing these conjugates cannot involve any operator t which transforms s_1 into s_2 and also s_2 into s_3, since the equation

$$s_1 s_2 = s_2 s_1$$

implies

$$t^{-1}s_1 s_2 t = t^{-1}s_2 s_1 t, \quad \text{or} \quad t^{-1}s_1 t \cdot t^{-1}s_2 t = t^{-1}s_2 t \cdot t^{-1}s_1 t.$$

Hence the following theorem has been proved: *If a complete set of conjugate operators under any group G is transformed according to a multiply transitive substitution group by the operators of G then either every possible pair of these conjugates is composed of commutative operators or no one of these conjugates is commutative with any operator of the set besides itself.*

§ 2. *Conjugate Sets Having Special Properties.*

When one operator of a set of conjugate operators under a group G is commutative with each of the others except one, or with only one of the others, then it is clearly possible to arrange all the operators of the set in distinct pairs which constitute systems of imprimitivity of the substitution group according to which the operators of the set are transformed under G. In particular, it results from this that if a complete set of conjugate operators under a group is composed of an odd number of operators it is not possible that one operator be commutative with each of the others save one, or that one of these operators be commutative with one and only one other operator of the set.

If a complete set of conjugate operators under G contains two and only two operators which are not commutative with some one of these conjugates then these two operators must be commutative with each other unless the set of conjugates is transformed under G according to an imprimitive group having three elements in a system of imprimitivity. In particular, if the number of operators in a complete set of conjugates under G is not divisible by 3 and if this set involves two and only two operators which are not commutative with a certain other operator of the set, then these two operators must be commutative with each other. As a more special case we may note the following theorem: *If exactly $p - 2$, p being a*

prime number > 3, *of the operators in a complete set of* p *conjugate operators are commutative with some one of these conjugates the remaining two operators of the set must be commutative with each other.*

With slight modifications the arguments of the preceding paragraph apply to the case when in a complete set of conjugate operators under a group one of these conjugates is commutative with two and only two of the others. If these two operators are commutative with each other, the total set of conjugates is clearly transformed according to an imprimitive substitution group, having three elements in a system, by all the operators of the group in question. Hence it results that in a set of p conjugates under the powers of an operator of the prime order p it is not possible that one of these conjugates is commutative with two and only two of the others if these two are commutative and $p > 3$. In particular if a set of 5 conjugates under the powers of an operator of order 5 includes an operator which is commutative with at least one of the others without being commutative with each of them, then this operator is commutative with exactly two other operators and these two operators are non-commutative while the remaining two operators must be commutative.

If t transforms s into an operator which is commutative with s, then t^{-1} must evidently also transform s into an operator which is commutative with s. Hence *a complete set of conjugate operators under the powers of an operator of odd order must always involve an odd number of operators which are commutative with a particular operator of the set.* The preceding theorems relating to a complete set of conjugate operators clearly apply also to a complete set of conjugate subgroups if we replace the term commutative by the term permutable.

§ 3. *Determination of the Possible Groups Generated by Two Operators of the Same Prime Order such that the Conjugates of One under the Other are Commutative.*

Let s_1 and s_2 be two distinct operators of the same prime order p, and suppose that the p conjugates of s_2 under the powers of s_1 are commutative with each other. There are two possible cases to be considered. In the first case these p conjugates of s_2 are identical and the group generated by s_1 and s_2 is either the group of order p generated by s_2 or the

non-cyclic group of order p^2, as s_1 generates or does not generate s_2. This case is practically trivial and requires no further consideration.

In the second possible case the p conjugates of s_2 under the powers of s_1 are distinct. In this case the order of the group generated by s_1 and s_2 cannot be less than p^3 nor greater than p^{p+1}, since the p conjugates in question cannot generate a group whose order is less than p^2 or greater than p^p. When $p = 2$, the upper and lower limits for the orders of the possible groups are equal, and hence the octic group is the only possible group in this case. In general, *for every prime number p there are $p - 1$ groups which are generated by two non-commutative operators of order p satisfying the condition that the p conjugates of one of these operators under the powers of the other are commutative.*

Since the conjugates of s_2 under the powers of s_1 are commutative, it results that the powers of s_2s_1 may be represented as follows, if $s_1s_2s_1^{-1} = t_1s_2$, and $s_1t_as_1^{-1} = t_{a+1}t_a$:

$$s_2s_1 = s_2s_1, \quad (s_2s_1)^2 = s_2s_1s_2s_1^{-1}s_1^2 = t_1s_2^2s_1^2, \quad (s_2s_1)^3 = t_2t_1^3s_2^3s_1^3,$$

$$(s_2s_1)^n = t_{n-1}t_{n-2}^n \cdots t_{n-r}^{\frac{n(n-1)\cdots(n-r+2)}{(r-1)!}} \cdots s_2^ns_1^n$$

$$(r = 2, 3, \cdots, n-1).$$

If we replace n by p in this formula it results directly that the order of s_2s_1 is p whenever t_{n-1} is identity, that is, whenever the order of the commutative commutator subgroup of the group G generated by s_1, s_2 has less than $p - 1$ independent generators. Since the commutator subgroup of G is always of index p^2 it results that whenever the order of G is less than p^{p+1} it cannot contain any operator of order p^2, and that G always contains exactly $p + 1$ subgroups of index p.

When G is of maximal order, two of its subgroups of index p involve no operators of order p^2, but in the remaining $p - 2$ of these subgroups all the operators are of order p^2 except those found in their common subgroup, which is the commutator subgroup of G. Hence each of these groups of maximal order is generated by its operators of order p, and whenever $p > 2$ each of these groups is also generated by its operators of order p^2. In the special case when $p = 2$, these operators clearly generate only the cyclic subgroup of order 4.

The system of groups under consideration is of special interest because it includes $p - 2$ non-abelian groups in which every operator except identity is of order p, for every possible value of the prime number p. The number of these groups for a particular prime number therefore depends upon this prime. The fact that the group generated by s_1, s_2 is completely determined by its order seems also worth noting.

UNIVERSITY OF ILLINOIS,
December, 1916.

A THEOREM IN THE ANALYSIS OF REAL VARIABLES.

BY PROFESSOR R. L. BORGER.

(Read before the American Mathematical Society, April 21, 1916.)

IN this paper the following theorem is proved:

THEOREM: *If the two real functions $U(x, y)$, $V(x, y)$ of the real variables x, y satisfy the following conditions at each point of a closed region R:*

(*a*) *U and V continuous in x and y jointly;*

(*b*) $$\frac{\partial U}{\partial x} \equiv U_1, \quad \frac{\partial U}{\partial y} \equiv U_2, \quad \frac{\partial V}{\partial x} \equiv V_1, \quad \frac{\partial V}{\partial y} \equiv V_2$$

exist and are finite;

(*c*) $\Delta U = hU_1 + kU_2 + \rho_1(h, k), \quad \Delta V = hV_1 + kV_2 + \rho_2(h, k);$

(*d*) $\lim\limits_{h,\, k \doteq 0} \dfrac{|\rho_i(h, k)|}{|h| + |k|} = 0 \quad (i = 1, 2);$

(*e*) $U_1 = V_2, \quad U_2 = -V_1;$

then U and V are analytic functions of x, y in R.

An immediate consequence of the theorem is that if any function W of a complex variable z possesses a finite derivative at each point of a simply connected closed region R then:

1. This derivative is continuous.
2. All the derivatives of W exist.
3. The function W may be represented by a power series in z.

To prove the theorem we apply the following, due to Kowalewski:*

If the two real functions $U(x, y)$, $V(x, y)$ are properly differentiable (satisfy hypotheses c, d) and if

$$U_1 = V_2$$

at each point of a rectangle R, there exists a function $\bar{U}(x, y)$ such that

$$\bar{U}_1 = U; \quad \bar{U}_2 = V.$$

The function U is, moreover, properly differentiable.

Employing this theorem we may infer the existence of two functions (in view of both parts of hypotheses (e)) $\bar{U}$, $\bar{V}$ such that

$$(1) \qquad \bar{U}_1 = U, \quad \bar{U}_2 = -V, \quad \bar{V}_1 = V, \quad \bar{V}_2 = U.$$

From equations (1)

$$(2) \qquad \bar{U}_1 = \bar{V}_2; \quad \bar{U}_2 = -\bar{V}_1.$$

Since $\bar{U}$ and $\bar{V}$ are properly differentiable we have by a second application of Kowalewski's theorem, two other functions, $\bar{\bar{U}}$, $\bar{\bar{V}}$, also properly differentiable and such that

$$(3) \qquad \bar{\bar{U}}_1 = \bar{U}; \quad \bar{\bar{U}}_2 = -\bar{V}; \quad \bar{\bar{V}}_1 = \bar{V}; \quad \bar{\bar{V}}_2 = \bar{U}.$$

The functions $\bar{\bar{U}}(x, y)$, $\bar{\bar{V}}(x, y)$, as well as their first and second partial derivatives, are easily seen to be continuous in the variables x, y and satisfy Laplace's differential equation $\Delta U = 0$, since

$$(4) \qquad \begin{aligned} \frac{\partial^2 \bar{\bar{U}}}{\partial x^2} &= \frac{\partial \bar{\bar{U}}_1}{\partial x} = \frac{\partial \bar{U}}{\partial x} = \bar{U}_1 = U, \\ \frac{\partial^2 \bar{\bar{U}}}{\partial y^2} &= \frac{\partial \bar{\bar{U}}_2}{\partial y} = -\frac{\partial \bar{V}}{\partial y} = -V_2 = -U. \end{aligned}$$

$$(5) \qquad \therefore \quad \frac{\partial^2 \bar{\bar{U}}}{\partial x^2} + \frac{\partial^2 \bar{\bar{U}}}{\partial y^2} = 0.$$

Similarly for $\bar{\bar{V}}$.

The functions $\bar{\bar{U}}$, $\bar{\bar{V}}$ are then analytic† functions of x, y.

* Kowalewski, Die komplexen Veränderlichen und ihre Funktionen, p. 187.

† Picard, Traité d'Analyse, tome 2 (2d ed.), p. 18.

The functions U and V being contained among the derivatives of $\bar{U}$ and $\bar{V}$ are also analytic functions of x, y.

It may now be seen that we have the following representation for these functions:

$$(6)\qquad \begin{aligned} U(x, y) &= \sum_{k=0}^{\infty} (a_{10}x + a_{01}y)^k; \\ V(x, y) &= \sum_{k=0}^{\infty} (b_{10}x + b_{01}y)^k; \end{aligned}$$

$$((a_{10})^l(a_{01})^m = a_{lm};\quad (b_{10})^l(b_{01})^m = b_{lm};\quad l, m = 0, \cdots, k).$$

Differentiating the equations

$$U_1 = V_2,\quad U_2 = -V_1$$

$l - 1$ times with respect to x and $k - l$ times with respect to y, at the point (0, 0) we have the following relations among the coefficients:

$$(7)\quad a_{l,\,k-l} = b_{l-1,\,k-l+1};\quad b_{l,\,k-l} = -a_{l-1,\,k-l+1}\quad (l = 1, \cdots, k).$$

We may now prove that if a function W of a complex variable z possesses a finite derivative at each point of a simply connected closed region, R,

1. This derivative is continuous.
2. All the derivatives of W exist.
3. The function W may be represented as a power series in z.

Since U and V are analytic functions of x and y, we see immediately from the representation

$$W = U(x, y) + iV(x, y)$$

that 1 and 2 are true. From equations (6)

$$\begin{aligned} W = U(x, y) + iV(x, y) &= \sum_{k=0}^{\infty} (a_{10}x + a_{01}y)^k + i\sum_{k=0}^{\infty} (b_{10}x + b_{01}y)^k \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{k}{l} [a_{l,\,k-l} + ib_{l,\,k-l}]x^l \cdot y^{k-l}. \end{aligned}$$

From equations (7) we have

$$a_{l-1,\,k-l+1} + ib_{l-1,\,k-l+1} = i(a_{l,\,k-l} + ib_{l,\,k-l})\quad (l = 1, \cdots, k)$$

and in general

$$a_{l-j,\,k-l+j} + ib_{l-j,\,k-l+j} = (i)^j(a_{l,\,k-l} + ib_{l,\,k-l})\quad (j = 1, \cdots, l).$$

For $l = k$ it follows that

$$a_{k-j,\,j} + ib_{k-j,\,j} = (i)^j(a_{k0} + ib_{k0}) \quad (j = 1, \cdots, k).$$

Then

$$W(z) = \sum_{k=0}^{\infty} (a_{k0} + ib_{k0})(x + iy)^k$$

$$= \sum_{k=0}^{\infty} C_k z^k, \text{ where } (C_k \equiv a_{k0} + ib_{k0}).$$

This completes the theorem.

I am indebted to Professor E. J. Townsend for suggesting the problem.

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CONCERNING THE COMPLEMENT OF A COUNTABLE INFINITY OF POINT SETS OF A CERTAIN TYPE.

BY DR. J. R. KLINE.

(Read before the American Mathematical Society, December 27, 1916.)

IN his "Grundzüge der Mengenlehre," Hausdorff proved that if E denotes a euclidean space of two or more dimensions while R is a countable set of points belonging to E, then $E-R$ is a connected* point set.† It is the object of the present paper to prove a theorem, which contains Hausdorff's theorem as a special case. Hausdorff's method of proof does not apply for the proof of the more general theorem. While the proof is carried out for the case of two dimensions, it is evident that a similar proof would apply to any higher number of dimensions.

THEOREM. *If M is a domain‡ and $G_1, G_2, G_3, \cdots$ is a countable infinity of nowhere dense§ closed point sets, no one of*

* A set of points is said to be *connected* if, however it be divided into two mutually exclusive proper subsets, one of them contains a limit point of the other.

† Cf. F. Hausdorff, "Grundzüge der Mengenlehre," Leipzig, Veit, 1914, p. 333.

‡ A domain is a connected set of points M, such that if P is a point of M, then there exists a region containing P and lying in M.

§ A set of points is said to be *nowhere dense*, if in every region R_1 there exists a region R_2 entirely free of points of the set. A set is said to be *closed* if it contains all its limit points.

which disconnects any domain, then $M-(G_1+G_2+G_3+\cdots)$ *is connected.*

*Proof.**—Let A and B denote any two distinct points of $M-(G_1+G_2+\cdots)$.† The point set $M-G_1$ is a domain. For each point P of $M-G_1$, let K_{1P} denote a region, of subscript greater than or equal to 1, which belongs to the fundamental sequence‡ and is such that K'_{1P}§ is a subset of $M-G_1$. Call the set of all such regions S_1. There exists a simple chain‖ $R_{11}, R_{12}, R_{13}, \cdots, R_{1n_1}$ from A to B every link of which is a region of the set S_1. Call this chain C_1. As two regions which have a point in common, also have in common a region containing that point, and as G_2 is a nowhere dense closed point set, it follows that R_{1i} and R_{1i+1} $(i=1, 2, \cdots, n_1-1)$ have in common a point P_{1i}, not belonging to G_2. Call A, P_{10} and B, P_{1n_1}. For each point P of R_{1i} $(i=1,2,3,\cdots,n_1)$ which does not belong to G_2, let K_{2P} denote a region of the set $K_2, K_3, \cdots$, which is such that K'_{2P} lies in R_{1i} and contains no point of G_2. Call the set of all such regions S_2. It is now possible to construct a simple chain C_2, which satisfies all the requirements of Professor Moore's C_2¶ and has the additional property that every link of C_2 belongs to S_2 and therefore contains no points of G_1+G_2. Continue this process. We obtain an infinite sequence of chains $C_1, C_2, C_3, \cdots$ which satisfy all the requirements of Moore's sequence of chains $C_1, C_2, C_3, \cdots$ and have the additional property that no link of the chain C_n has a point in common with the set G_1+G_2

* Our theorem is proved on the basis of the system of axioms Σ_1 proposed by R. L. Moore in his paper, "On the foundations of plane analysis situs," *Transactions Amer. Math. Soc.*, vol. 17 (1916), pp. 131–164.

† That $M-(G_1+G_2+G_3+\cdots)$ contains infinitely many points follows at once from one of the theorems of Baire, *Annali di Mat.* (3), vol. 3, p. 65.

‡ Select once for all a definite sequence, $K_1, K_2, \cdots$ satisfying the conditions of Axiom 1 of Σ_1. This definite sequence will be called the *fundamental sequence* and its regions will be termed *fundamental regions*.

§ The *boundary* of a point set M is the set of all limit points of M, which do not belong to M. If R is a region, R' denotes the point set composed of R plus its boundary.

‖ If A and B are distinct points, then a *simple chain* from A to B is defined by R. L. Moore as a finite sequence of regions $R_1, R_2, R_3, \cdots, R_n$ such that (1) R_i contains A if and only if $i=1$, (2) R_i contains B if and only if $i=n$, (3) if $1\leqq i\leqq n$ and $1\leqq j\leqq n$, while $i<j$, then R_i has a point in common with R_j if and only if $j=i+1$. The region R_k $(1\leqq k\leqq n)$ is said to be the kth *link* of the chain. See R. L. Moore, loc. cit., p. 134. For a proof of the existence of such a chain, see R. L. Moore, loc. cit., Theorem 10, p. 135.

¶ Cf. R. L. Moore, loc. cit, p. 137.

$+ \cdots + G_n$. Let $\overline{C}_n$ denote the point set which is the sum of all the links of the chain C_n, while C denotes the set of all points that the sets $\overline{C}_1, \overline{C}_2, \overline{C}_3, \cdots$ have in common. The point set C is a simple continuous arc* from A to B, lying entirely in the set $M - (G_1 + G_2 + \cdots)$.†

It follows that $M - (G_1 + G_2 + \cdots)$ is connected.

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AN ANALOGUE TO PASCAL'S THEOREM.

BY DR. A. L. MILLER.

A DECAGON is said to be doubly inscribed in a cubic if every odd side of the decagon cuts three even sides on the cubic and and every even side cuts three odd sides on the cubic.

That there exist decagons doubly inscribed in a cubic can be seen as follows. Let the decagon D have for sides $s_1, s_2, s_3, \cdots, s_{10}$ and let the cubic be C_3. Let

$$s_1 \text{ meet } s_{10}, s_2, s_4 \text{ on } C_3,$$

$$s_3 \text{ meet } s_2, s_4, s_6 \text{ on } C_3,$$

$$s_5 \text{ meet } s_4, s_6, s_8 \text{ on } C_3,$$

$$s_7 \text{ meet } s_6, s_8, s_{10} \text{ on } C_3,$$

while s_9 is the line joining the third intersection of s_8 with C_3 with the third intersection of s_{10} with C_3. Then, by Cayley's‡ theorem, s_9 also cuts s_2 on C_3.

By a repetition of this last theorem we obtain the following theorem analogous to Pascal's theorem:

If a decagon be doubly inscribed in a cubic the remaining ten intersections of the odd sides with the even ones lie on a conic.

UNIVERSITY OF MICHIGAN.

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* If A and B are distinct points, a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B, but containing no proper connected subset containing both A and B. See N. J. Lennes, "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, vol. 33 (1911) and this BULLETIN, vol. 12 (1906), p. 284.

† For a proof of this statement, see the proof of Theorem 15 of Moore's paper, loc. cit., pp. 136–9.

‡ Cayley: *Cambridge and Dublin Mathematical Journal*, vol. 3.

NOTES.

THE opening (January) number of volume 18 of the *Transactions of the American Mathematical Society* contains the following papers: "On the consistency and equivalence of certain definitions of summability," by W. A. HURWITZ and L. L. SILVERMAN; "The resolution into partial fractions of the reciprocal of an entire function of genus zero," by J. F. RITT; "On a general class of linear homogeneous differential equations of infinite order with constant coefficients," by J. F. RITT; "On the expressibility of a uniform function of several complex variables as the quotient of two functions of entire character," by T. H. GRONWALL; "Certain formal invariances in boolean algebras," by N. WIENER; "On a theory of linear differential equations in general analysis," by T. H. HILDEBRANDT; "Transformations T of conjugate systems of curves on a surface," by L. P. EISENHART.

Professor L. E. DICKSON has retired from the Editorial Committee, his unexpired term being filled by Professor L. P. EISENHART. Professors G. A. BLISS and E. B. WILSON have retired from the editorial staff and Professors C. N. MOORE and F. R. SHARPE have been appointed associate editors.

THE annual meeting of the London mathematical society was held November 2; Professor H. H. MACDONALD was elected president; the other officers were re-elected. The following papers were read at this meeting: by J. LARMOR, "Presidential address," and "Fourier harmonic analysis, its practical scope"; by W. H. YOUNG, "Multiple integration by parts and the second theorem of the mean"; by E. H. NEVILLE, "Moving axes and their uses in the differential geometry of euclidean space"; by J. H. HODGKINSON, "Areas and conformal representation."

AT the meeting held December 14 the following papers were presented: by W. BURNSIDE, "The efficiency of a surface of discontinuity regarded as a propeller"; by G. H. HARDY and S. RAMANUJAN, "Proof that almost all numbers are composed of about log log N prime factors," and "An asymptotic formula for the number of partitions of a number"; by G. N. WATSON, "The harmonic functions associated with a parabolic

cylinder (second paper)"; by D. BUCHANAN, "Orbits asymptotic to an isosceles triangle solution of the problem of three bodies"; by H. S. CARSLAW, "Diffraction of waves by a wedge of any angle"; by L. J. ROGERS, "Two theorems of combinatory analysis, and two allied identities"; by W. H. YOUNG and Mrs. YOUNG, "The internal structure of a set of points in space of any number of dimensions" and "The inherently crystalline structure of a function of any number of variables."

AT the meeting of the Edinburgh mathematical society on January 12 the following papers were read: by W. P. MILNE: "The apolar locus of two tetrads of points"; by L. R. FORD: "A method of solving algebraic equations."

AT the annual meeting of the British mathematical association held in London on January 5 the following papers were read: by T. P. NUNN, "The school syllabus in geometry"; by A. W. SIDDONS, "Some of the work of the teaching committee"; by A. N. WHITEHEAD, "Technical education and its relations to literature and science"; by A. W. SIDDONS, "An accuracy test set in some public schools"; by P. ABBOTT, "The place of mathematics in education reconstruction." Professor WHITEHEAD was elected president for the current year.

UNIVERSITY OF CHICAGO.—The following mathematical courses are announced for the summer quarter, June 18–August 31:—By Professor E. H. MOORE: The spectrum of limited infinite hermitian matrix (first half), four hours; Series (first half), four hours.—By Professor H. E. SLAUGHT: Differential equations, four hours.—By Professor E. J. WILCZYNSKI: Projective differential geometry, four hours.—By Professor J. W. A. YOUNG: Solid analytics, four hours.—By Professor A. C. LUNN: Relativity, four hours; Functions of a complex variable, four hours.—By Professor D. N. LEHMER (University of California): Synthetic projective geometry, four hours.—By Professor G. D. BIRKHOFF (Harvard University): Ordinary differential equations of the second order, four hours.

THE firm of G. E. Stechert and Company expect soon to have on sale a photographic reprint of the third edition of the

first volume of VALLÉE-POUSSIN's Cour d'Analyse. The book, in its original form, is extremely rare, very few copies having been sent out from Louvain before the destruction of the city.

PROFESSOR G. FUBINI, of the technical school at Turin, has been elected a corresponding member of the Accademia dei Lincei.

PROFESSOR E. LAURA, of the University of Messina, has been appointed associate professor of mechanics at the University of Padua.

DR. B. CALDONAZZO has been appointed docent in mechanics in the technical school at Milan.

MR. R. W. BARNARD has been appointed instructor in mathematics in the University of Michigan.

CHARLES J. WHITE, emeritus professor of mathematics in Harvard University, died February 12 at the age of 78 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOREL (E.). See VALLÉE-POUSSIN (C. DE LA).

CREMONA (L.). Opere matematiche. Tomo 3 con notizie della vita e delle opere dell'autore e con indice alfabetico per materie. Milano, Hoepli (Pisa, succ. fratelli Nistri), 1916. 4to. 22+520 pp. L. 30.00

FRAENKEL (A.). Ueber gewisse Teilbereiche und Erweiterungen von Ringen. Leipzig, Teubner, 1916. 8vo. 64 pp. M. 2.80

HANCOCK (H.). Elliptic integrals. (Mathematical monographs, No. 18.) New York, Wiley, 1916. 8vo. 125 pp. $1.25

JOSEPH (H. W. B.). An introduction to logic. 2d edition, revised and enlarged. New York, Oxford University Press, 1916. 8vo. 12+608 pp. $3.15

JOYCE (G. H.). Principles of logic. 2d edition. New York, Longmans, 1916. 8vo. 20+431 pp. $2.50

MANCINI (G.). L'opera "De corporibus regularibus" di Pietro Franceschi detto Della Francesca ursurpata da Fra Luca Pacioli. Memoria con tavole. (*Reale Accademia dei Lincei*, Anno CCCXII, 1915.) Roma, Pio Befani, 1915. 4to. 144 pp.+17 tavole.

PHILLIPS (H. B.). Differential calculus. New York, Wiley, 1916. 8vo. 5+162 pp. $1.25

PRINGSHEIM (A.). Vorlesungen über Zahlen- und Funktionenlehre. 1ter Band, 1te Abteilung: Reelle Zahlen und Zahlenfolgen. Leipzig, Teubner, 1916. 8vo. 12+292 pp. Geh. M. 12.00

SLAUGHT (H. E.). See WILCZYNSKI (E. J.).

STAGER (H. W.). A Sylow factor table of the first twelve thousand numbers, giving the possible number of Sylow sub-groups, of a group of given order between the limits of 0 and 12,000. Washington, Carnegie Institution, 1916. 12+120 pp.

VALLÉE-POUSSIN (C. DE LA). Intégrales de Lebesgue. Fonctions d'ensemble. Classes de Baire. Leçons professées au Collège de France. (Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Emile Borel.) Paris, Gauthier-Villars, 1916. 8vo. 8+154 pp. Fr. 7.00

WILCZYNSKI (E. J.). College algebra with applications edited by H. E. Slaught. Boston, Allyn and Bacon, 1916. 8vo. 20+507 pp. Cloth. $2.00

II. ELEMENTARY MATHEMATICS.

ARNOLD (E. E.). See DURELL (F.).

BETZ (W.) and WEBB (H. E.). Plane and solid geometry. With the editorial cooperation of P. F. Smith. Boston, Ginn, 1916. 11+507 pp. Cloth. $1.36

BOOKMAN (C. M.). Business arithmetic. New York, American Book Company, 1914. 12mo. 250+12 pp. $0.65

COFFMAN (L. D.). See JESSUP (W. A.).

DEMING (A. G.). Number stories. Chicago, Beckley-Cardy Company, 1916. 205 pp. Cloth.

DURELL (F.) and ARNOLD (E. E.). Plane geometry. New York, Merrill, 1916. 300 pp. $0.88

HAMM (F. P.). Outline and suggestive methods and devices on the teaching of elementary arithmetic. Philadelphia, Lippincott, 1916. 5+40 pp. $1.00

JESSUP (W. A.) and COFFMAN (L. D.). Supervision of arithmetic. New York, Macmillan, 1916. 7+225 pp. Cloth.

MILNE (W. J.). Second course in algebra. New York, American Book Company, 1914. 12mo. 288 pp. Cloth. $0.88

——. Standard algebra, revised. New York, American Book Company, 1914. 12mo. 496+52 pp. $1.00

RIVENBURG (R. H.). A review of algebra. New York, American Book Company, 1914. 12mo. Cloth. 80 pp. $0.36

ROBBINS (E. R.). New plane and solid geometry. New York, American Book Company, 1916. 8vo. Cloth. 460 pp. $1.40. Solid geometry. 158 pp. $0.80. Plane geometry. 264 pp. $0.80

SMITH (P. F.). See BETZ (W.).

TAVOLE logaritmiche a cinque cifre decimali, raccolte e pubblicate per cura dell'Istituto idrografico della r. marina. 2a edizione. Genova, tip. Istituto idrografico della r. marina, 1916. 8vo. 27+524 pp.

THOMAS (A. O.). Rural arithmetic. New York, American Book Company, 1916. 8vo. 288 pp. Cloth. $0.68

WEBB (H. E.). See BETZ (W.).

WHEELER (A. H.). Examples in algebra. Boston, Little, Brown and Company, 1916. 14+257 pp. $0.90

III. APPLIED MATHEMATICS.

ARPESANI (C.). Elementi di tecnologia meccanica. Lavorazione dei metalli. 2a edizione rinuovata ed ampliata. Milano, Hoepli, 1915. 16+603 pp. L. 5.50

AUSTIN (F. E.). Examples in alternating currents. (For students and engineers.) Vol. 1. 2d edition. Hanover, N. H., F. E. Austin, 1916. 223 pp. Cloth. $2.00

BELDING (A. G.). Accounts and accounting practice. New York, American Book Company, 1916. 8vo. 224 pp. $0.90

BEXELL (J. A.) and NICHOLS (F. G.). Principles of book-keeping and farm accounts. New York, American Book Company, 1914. 8vo. 180 pp. $0.65

BRODIE (O. L.). See MORRISON (C. E.).

BROUGH (B. H.). A treatise on mine-surveying. 14th edition revised and enlarged by H. Dean. London, C. Griffin, 1916. 18+477 pp. 7s. 6d.

CAIN (W.). Earth pressure, retaining walls and bins. New York, Wiley, 1916. 297 pp. Cloth. $2.50

CARPENTER (C. K.). See WOLFARD (M. R.).

CLAUDI (C.). Manuale di prospettiva. 4a edizione riveduta. Milano, Hoepli, 1915. 12+76 pp. L. 2.50

CONTALDI (P.). La meccanica e le macchine nella scuola e nell'industria. 3a edizione riveduta ed ampliata. Milano, Hoepli, 1914–1916. 8vo. 16+676+16+768 pp. L. 12.00+12.00

DEAN (H.). See BROUGH (B. H.).

DEL FABRO (G.). Manuale di topografia per pratica e per studio. 3a edizione riveduta ed aumentata. Milano, Hoepli, 1914. 43+629 pp.

EFFEMERIDI astronomiche ad uso dei naviganti per l'anno 1917 (Istituto idrografico della r. marina). Genova, tip. Istituto idrografico della r. marina, 1916. 8vo. 187 pp. L. 1.50

FORBES (G.). David Gill: man and astronomer. Memories of Sir David Gill, astronomer (1879–1907) at Cape of Good Hope. Collected and arranged by G. Forbes. London, Murray, 1916. 11+418 pp. 12s.

GARUFFA (E.). L'aviazione. Milano, Hoepli, 1916. L. 8.50

HOLT (A. H.). Field astronomy. New York, Wiley, 1917. 8vo. 10+128 pp. Leather. $1.50

KAYE (G. W. C.) and LABY (T. H.). Tables of physical and chemical constants and some mathematical functions. 2d edition. New York, Longmans, 1916. 8vo. 8+153 pp. $2.00

KIDDER (F. E.) and NOLAN (T.). Architects' and builders' pocket book. 16th edition, rewritten. New York, Wiley, 1915. 1856 pp. $5.00

LABY (T. H.). See KAYE (G. W. C.).

LORENTZ (H. A.). The theory of electrons and its applications to the phenomena of light and radiant heat. 2d edition. Leipzig, Teubner, and New York, Stechert, 1916. 8vo. Geh. M. 9.00

MALAVASI (C.). Vademecum per l'ingegnere costruttore meccanico. 3a edizione notevolmente ampliata, con speciale riguardo alle esigenze dell'insegnamento tecnico. Milano, Hoelpi, 1916. 36+862 pp. L. 10.50

MARGUET (F.). Cours d'astronomie. Paris, Challamel, 1916. 8vo. Fr. 10.00

MIESSNER (B. F.). Radiodynamics. The wireless control of torpedoes and other mechanisms. New York, Van Nostrand, 1916. 206 pp. Cloth. $2.00

MORRISON (C. E.) and BRODIE (O. L.). Masonry dam design, including high masonry dams. 2d edition. New York, Wiley, 1916. 285 pp. Cloth. $2.50

NICHOLS (F. G.). See BEXELL (J. A.).

NOLAN (T.). See KIDDER (F. E.).

RAYMOND (W. G.). Plane surveying. New York, American Book Company, 1914. 12mo. 589 pp.+7 plates. Leather. $3.00

——. Railroad field manual for civil engineers. New York, Wiley, 1915. 7+398 pp. Leather. $3.00

RONCHETTI (G.). Grammatica del disegno. Metodo pratico per imparare il disegno. 2a edizione, interamente rifatta. Milano, Hoepli, 1915. 8vo. 8+225 pp.+un vol. di 80 tavole. L. 7.50

RORAY (N. L.). Industrial arithmetic. Philadelphia, Blakiston, 1916. 8+144 pp. $0.75

SPATARO (D.). Trattato teorico e pratico di idromeccanica. Volume 1. Milano, Hoepli, 1915. 8vo. 16+448 pp. L. 10.00

STEVENS (J. S.). Theory of measurements. A manual for physics students. New York, Van Nostrand, 1915. 8vo. 88 pp. $1.25

VIAPPIANI (A.). Trattato di idraulica pratica. Raccolta di formole e dati pratici da servire di guida nello studio nelle questioni relative al movimento delle acque, sia per utilizzarle in pro dell' agricoltura, industria, igiene e navigazione. . . . 2a edizione, riveduta e sensibilmente riformata. Milano, Hoepli, 1915. 32+740 pp.+14 tavole. L. 14.00

WOLFARD (M. R.) and CARPENTER (C. K.). The W-PVT chart. New York, Wiley, 1915. 24 by 38 inches. $0.50

LIBRARY.

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-ninth regular meeting of the Society was held in New York City on Saturday, February 24. The morning session sufficed for the presentation of the brief list of papers. The attendance included the following twenty-six members:

Mr. D. R. Belcher, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor C. C. Grove, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. R. Kline, Mr. Harry Langman, Professor R. L. Moore, Mr. G. W. Mullins, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor W. B. Stone, Mr. H. E. Webb, Professor H. S. White, Mr. J. K. Whittemore.

Professor H. S. White occupied the chair, being relieved by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Professor H. P. Kean, McHenry College; Mr. Ralph Keffer, Harvard University; Mr. H. C. M. Morse, Harvard University; Dr. F. D. Murnaghan, Rice Institute; Mr. G. E. Raynor, University of Washington; Dr. S. P. Shugert, University of Pennsylvania; Mr. G. W. Smith, University of Illinois; Mr. J. S. Taylor, University of California; Dr. L. E. Wear, University of Washington; Dr. H. N. Wright, University of California. Four applications for membership in the Society were received.

It was decided to hold the next summer meeting of the Society at Cleveland, Ohio, on September 4–5. The Mathematical Association of America will meet at Cleveland on September 6–7. The two organizations have appointed a joint committee on arrangements for the meetings, consisting of Professors Focke, Huntington, Pitcher, D. T. Wilson, and Secretaries Cole and Cairns.

At the annual meeting of the Society, the Council placed itself on record as desiring to cooperate with the National Research Council in forwarding the interests of research. At the February meeting, a committee was appointed to confer with

the chairman of the Mathematics Committee of the Research Council, Professor E. H. Moore, in regard to the selection of the members of that committee.

By the will of the late Professor L. L. Conant, who was a member of the Society from 1892 to 1916, the sum of $10,000 is left to the Society, subject to Mrs. Conant's life interest. The will provides that the income of this bequest "shall be offered once in five years as a prize for original work in pure mathematics." This generous gift, a noble monument to the donor, should do much for the promotion of higher mathematical aims in this country. For many years the Society has consistently pursued these aims, with a success far outrunning what might have been expected from its modest financial resources. With greater means in the way of general or special funds, it could accomplish still more. To anyone who is able to give for science, the Society presents itself as an experienced and beneficent administrator.

The following papers were read at the February meeting:

(1) Dr. A. R. SCHWEITZER: "The iterative compositions of a function of $n + 1$ variables ($n = 1, 2, 3, \cdots$)."

(2) Dr. A. R. SCHWEITZER: "Functional equations based on iterative compositions."

(3) Dr. D. F. BARROW: "An application of Fourier's series to probability."

(4) Professor EDWARD KASNER: "Degenerate cases in the theory of conduction of heat."

(5) Professor J. E. ROWE: "The equation of a rational plane curve derived from its parametric equations (second paper)."

(6) Professor R. L. MOORE and Dr. J. R. KLINE: "The most general closed plane point set through which it is possible to pass a simple continuous plane arc."

(7) Professor H. S. WHITE: "New proof of a theorem of von Staudt and Hurwitz."

(8) Professor HENRY TABER: "On the structure of finite continuous groups."

In the absence of the authors, the papers of Dr. Schweitzer, Dr. Barrow, Professor Rowe, and Professor Taber were read by title. Abstracts of the papers follow below.

1. Dr. Schweitzer constructs an exhaustive table of iterative compositions of a function of $n + 1$ variables ($n = 1, 2, 3, \cdots$). A symbolic representation for such a composition

is used, and by introducing a simple convention the compositions are readily ordered for each value of n. The construction of the table is based on the general principle implied by the following operation: If the function $f(x_0, x_1, \cdots, x_n)$ has some one of its arguments, say x_0, replaced by the function f, then this f is denoted by $f(x_{00}, x_{01}, \cdots, x_{0n})$ and the resulting composition is denoted by the symbol

$$[01, \cdots, n; (00, 01, \cdots, 0n), 1, \cdots, n].$$

More compactly, the author puts, e. g., $F(01) \equiv f(x_0, x_1)$, $F(01; 1) \equiv [F(01); 0, (10, 11)]$, etc. Similarly for three variables one obtains $F(012)$, $F(012; 1)$, etc., $F(012; 11)$, etc. Any iterative composition of a function of $n + 1$ variables is represented, then, by the symbol $F(012 \cdots n; i_1, i_2, \cdots)$ where i_1, i_2, etc., are certain integers. To illustrate, for $n = 1$ the iterative compositions are $f(x_0, x_1)$, $f\{x_0, f(x_{10}, x_{11})\}$, $f\{f(x_{00}, x_{01}), x_1\}$, $f\{f(x_{00}, x_{01}), f(x_{10}, x_{11})\}$, $f\{x_0 f[x_{10}, f(x_{110}, x_{111})]\}$, etc.

2. In Dr. Schweitzer's second paper a systematic application is made of the preceding iterative compositions of a function of $n + 1$ variables ($n = 1, 2, 3, \cdots$) to the genesis of functional equations, by means of the inversion, elimination, or iteration of some or all of the involved variables (the processes being taken separately or in combination). The author obtains three categories of functional equations, viz., the inversive, the eliminative, and the iterative, which, however, are not necessarily mutually exclusive. It is found that certain eliminative functional equations, including certain of those called "quasi-transitive," are implied by certain purely inversive functional relations. An important class of inversive functional equations is generated by the following requirement: Given the iterative composition $F(012 \cdots n; i_1, i_2, \cdots)$ and a substitution group G on the variables involved (leaving some or none of them fixed); to find the function $f\{x_1, x_2, \cdots, x_{n+1}\}$ which satisfies the functional equations expressed by the formal invariance of the composition $F(012 \cdots n; i_1, i_2, \cdots)$ under the substitutions of the group G. On the other hand extensive new categories of eliminative functional equations are obtained. A few of the theorems proved are as follows:

I. If $f\{f(x_1, x_2), f(x_3, x_4)\}$ is formally invariant under the cyclic group on x_1, x_2, x_3, x_4 then $f(x_1, x_2) = x^{-1}\{cx(x_1) + cx(x_2)\}$ where c is an arbitrary constant.

II. This theorem is a direct generalization of theorem I and incidentally generalizes Hayashi's generalization of Abel's functional equations. Instead of the symmetric group the cyclic group is used.

III. If $f\{f(x_1, x_2), f(x_3, x_4)\}$ is formally invariant under the non-cyclic group of order four on x_1, x_2, x_3, x_4, and if $f(x, x) = \text{const.}$, $f\{f(x_1, x_2)c_1\} = \lambda f(x_1, x_2)$, then $f(x_1, x_2) = x^{-1}\{cx(x_1) - cx(x_2)\}$, where c is an arbitrary constant.

IV. If $f\{x, f(t_1, t_2, y), f(t_1, t_2, z)\} = f(x, y, z)$, then $f(x, y, z) = x^{-1}\{\vartheta(x) - x(y) + x(z)\}$, where $\vartheta(x)$ is arbitrary.

V. If $f\{f(x, y), f[x, f(x, \cdots, f\{x, z\})]\} = f(z, y)$, where in the latter equation the symbol f occurs $n + 3$ times $(n = 1, 2, 3, \cdots)$, then $f(x, y) = x^{-1}\{x(x) + wx(y)\}$, where w is a root of the equation $w^n + w^{n-1} + \cdots + w + 1 = 0$.

3. This paper is the outgrowth of an attempt on the part of Dr. Barrow to solve a problem proposed by Professor E. W. Brown. A solution of the problem and a new proof of certain well known properties of frequency curves are obtained as applications of the main theorem of the paper which is as follows:

Let two quantities N and M have frequencies $f(x)$ and $\varphi(x)$ respectively, and let $F(x)$ denote the frequency of $M + N$. If these three functions are identically zero outside some interval $(-C, C)$, and capable of development in Fourier's series within this interval, then, letting a_i and b_i, α_i and β_i, A_i and B_i denote the ith coefficients of the cosines and sines in these series, we have

$$A_i + B_i\sqrt{-1} = C(a_i + b_i\sqrt{-1})(\alpha_i + \beta_i\sqrt{-1}).$$

Equating the real and imaginary parts enables us to calculate the coefficients in the development of $F(x)$ from those in the developments of $f(x)$ and $\varphi(x)$.

4. The partial differential equation for the conduction of heat in a plane determines the temperature as a function of the position (x, y) and the time t. For each solution $\varphi(x, y, t)$ there will be, in general, ∞^2 isothermal curves $\varphi(x, y, t) = c$.

Professor Kasner finds all solutions for which the doubly infinite system degenerates into a simply infinite system. There are three types: In the first $\varphi = f(x, y)$, the temperature not entering; in the second, $\varphi = f(x, y) + t$; in the third, $\varphi = e^t f(x, y)$. The differential equations for these types are respectively $\Delta f = 0$, $\Delta f = 1$, $\Delta f = f$, where $\Delta f \equiv f_{xx} + f_{yy}$.

5. Professor Rowe's paper appears in full in the present number of the BULLETIN.

6. In order that it may be possible to pass a simple continuous arc through a given closed point set M it is sufficient that M should contain no connected subset consisting of more than one point. Cf. F. Riesz, *Comptes Rendus*, volume 141 (1905), page 650, A. Denjoy, ibid., volume 151 (1910), page 140, and L. Zoretti, ibid., volume 142 (1906), page 763. It is clear that this is not a necessary condition. Professor Moore and Dr. Kline have obtained the following theorem:

In order that it may be possible to pass at least one simple continuous arc through a given closed point set M it is necessary and sufficient that every closed connected subset of M should be either a single point or a simple continuous arc k no point of which, except its end points, is a limit point of $M - k$.

7. The theorem concerning two tetrahedra inscribed in a twisted cubic curve was announced by von Staudt in his Beiträge, and proved by Hurwitz thirty years later. Professor White points out that a simpler proof is found by projecting the curve, from all its points, into a plane. Then the theorem in question is derived easily from the well-known situation of two triangles inscribed to a conic, which is the Poncelet theorem for triangles.

8. In Professor Taber's paper $X_1, \cdots, X_r$ are the infinitesimal transformations of a finite continuous group G, in which case $(X_i, X_j) = \Sigma_1^r c_{ijk} X_k$ $(i, j = 1, 2, \cdots, r)$. Any infinitesimal transformation $\Sigma \alpha_i X_i$ of G can be represented by the point $\alpha = (\alpha_1, \cdots, \alpha_r)$ in the space of $r - 1$ dimensions upon which the group Γ adjoint to G operates. If now $f(\alpha) \equiv f(\alpha_1, \cdots, \alpha_r)$ is an invariant of Γ, it follows that the alternant of $\Sigma \alpha_i X_i$ and any other infinitesimal transformation of G is represented by

a point which lies in the polar $r-2$ flat of α qua $f(\alpha)=0$. Whence it follows that any $r-2$ flat $\Sigma u_i\alpha_i=0$ is invariant to each of the infinitesimal transformations of Γ represented by the poles of this flat qua $f(\alpha)=0$; and thus, if the poles of this flat do not all lie in an $r-2$ flat, it follows that the infinitesimal transformations of G represented by points in the $r-2$ flat $\Sigma u_i\alpha_i=0$ generate an invariant subgroup of G.

If the adjoint group Γ has a quadratic invariant of non-zero discriminant, for a proper choice of the X's, we shall have $c_{ijk}+c_{ikj}=0$ $(i, j, k=1, 2, \cdots, r)$. In this case the condition that $f(\alpha)$ shall be invariant to Γ is that

$$\Sigma\alpha_i E_i\left(\frac{\partial f(\alpha)}{\partial\alpha_1}, \cdots, \frac{\partial f(\alpha)}{\partial\alpha_r}\right)=0,$$

where E_i is the matrix whose constituent in the μth row and νth column is $c_{i\nu\mu}$ $(i, \mu, \nu=1, 2, \cdots, r)$; or, what is the same thing, is that

$$\left(\Sigma\alpha_i X_i, \Sigma\frac{\partial f(\alpha)}{\partial\alpha_i}X_i\right)=0$$

for all values of the α's. Whence it follows, if $f(\alpha)$ is a second invariant of Γ, that the infinitesimal transformation $\Sigma u_i X_i$ is commutative with every infinitesimal transformation of G represented by a pole, qua $f(\alpha)=0$, of the $r-2$ flat $\Sigma u_i\alpha_i=0$; and, if these poles do not all lie in any $r-2$ flat, it follows that $\Sigma u_i X_i$ is an exceptional infinitesimal transformation.

F. N. Cole,
Secretary.

THE EQUATION OF A RATIONAL PLANE CURVE DERIVED FROM ITS PARAMETRIC EQUATIONS (SECOND PAPER).

BY PROFESSOR J. E. ROWE.

(Read before the American Mathematical Society, February 24, 1917.)

As this is the second article on the same subject published by the author in the Bulletin, it is desirable to inform the reader at once that the method of deriving the equation of a rational plane curve from its parametric equations described in this paper is published not merely because it is a new

method but also because in many respects it is an improvement upon previous methods. To verify this statement the reader is referred to the processes given by Salmon (Higher Plane Curves, § 44), by Grace and Young (Algebra of Invariants, § 254), and to the method of my former article,* which was also developed independently by Mr. H. W. Richmond, of Cambridge University, but not published by him until after my article had appeared.

As in the preceding article, it seems best for the sake of readability to present the method through the instrumentality of the easiest illustrations.

First, let

$$\rho x_i = a_i t^2 + b_i t + c_i \qquad (i = 0, 1, 2) \tag{1}$$

be the parametric equations of a conic or R^2. By multiplying† each of these equations by k, transposing all terms to the same side, and replacing $-k\rho x_i$ by σx_i, we obtain

$$a_i k t^2 + b_i k t + c_i k + \sigma x_i = 0 \qquad (i = 0, 1, 2). \tag{2}$$

Solving (2) for the ratios of kt^2, kt, k, σ regarded as independent quantities, we have

$$kt^2 : -kt : k : -\sigma = |bcx| \ddagger : |acx| : |abx| : |abc|. \tag{3}$$

From (3)

$$-t = \frac{kt^2}{-kt} = \frac{-kt}{k} = \frac{|bcx|}{|acx|} = \frac{|acx|}{|abx|}. \tag{4}$$

Hence from (4)

$$\begin{aligned} |abx|\,t + |acx| &= 0, \\ |acx|\,t + |bcx| &= 0. \end{aligned} \tag{5}$$

Eliminating t we have

$$\begin{vmatrix} |abx| & |acx| \\ |acx| & |bcx| \end{vmatrix} = 0,$$

and this is the equation of the R^2 of (1). Obviously in this

* This BULLETIN, vol. 22, No. 7 (April, 1916), pp. 338–340. Also, see Professor Richmond's article, vol. 23, No. 2 (November, 1916), pp. 90–91.

† This is merely a matter of personal preference.

‡ By $|abx|$ is meant the three-rowed determinant $\begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix}$.

simple case the same result might have been obtained by simplifying the last equality in (4).

Further, let

$$\rho x_i = a_i t^3 + b_i t^2 + c_i t + d_i \qquad (i = 0, 1, 2) \tag{6}$$

be the parametric equations of the rational plane cubic R^3. Following (2), these may be written

$$a_i k t^3 + b_i k t^2 + c_i k t + d_i k + \sigma x_i = 0 \quad (i = 0, 1, 2). \tag{7}$$

These may be arranged in a form similar to (2) in a number of ways; for example,

$$a_i t \cdot kt^2 + b_i t \cdot kt + (c_i t + d_i)k + \sigma x_i = 0, \tag{8}$$

or

$$(a_i t + b_i)kt^2 + c_i \cdot kt + d_i k + \sigma x_i = 0. \tag{9}$$

Solving these equations for the ratios, we find

$$\begin{aligned} (10)\quad & kt^2 : -kt : k : -\sigma \\ & = (|bcx|t + |bdx|) : (|acx|t + |adx|) : |abx|t : U_3{}^* \\ (11)\quad & = |cdx| : (|adx|t + |bdx|) : (|acx|t + |bcx|) : U_3'. \end{aligned}$$

As in (4), it follows from (10) that

$$\begin{aligned} -t &= (|bcx|t + |bdx|)/(|acx|t + |adx|) \\ &= (|acx|t + |adx|)/|abx|t; \end{aligned} \tag{12}$$

similarly, from (11)

$$\begin{aligned} -t &= |cdx|/(|adx|t + |bdx|) \\ &= (|adx|t + |bdx|)/(|acx|t + |bcx|). \end{aligned} \tag{13}$$

Multiplying and transposing in (12) and (13) yields the system of equations

$$\begin{aligned} & |abx|t^2 + |acx|t + |adx| = 0, \\ (14)\quad & |acx|t^2 + (|adx| + |bcx|)t + |bdx| = 0, \\ & |adx|t^2 + |bdx|t + |cdx| = 0. \end{aligned}$$

The determinant of (14) equated to zero is the equation of the R^3.

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* U_3 and U_3' need not be calculated, as they are not used in subsequent work.

In deriving the equation of the R^n whose parametric equations are

$$a_i k t^n + b_i k t^{n-1} + \cdots + \sigma x_i = 0 \qquad (i = 0, 1, 2), \tag{15}$$

it is never necessary to solve for the ratios kt^2, kt, k, σ in more than $(n + 1)/2$ ways. The third equation of (14) might have been obtained from the first by interchanging a_i and d_i, b_i and c_i with the subsequent change of t to $1/t$. Obviously the second equation of (14) is invariant under this transformation. Further, if we make $n = 5$ in (15), the equation of the R^5 whose parametric equations are so obtained may be found by eliminating the powers of t from five equations. Three of these five equations

$$\begin{aligned}
&|abx|t^4 + |acx|t^3 + |adx|t^2 + |aex|t + |afx| = 0,\\
&|acx|t^4 + (|adx| + |bcx|)t^3 + (|aex| + |bdx|)t^2\\
&\qquad\qquad + (|afx| + |bex|)t + |bfx| = 0,\\
&|adx|t^4 + (|aex| + |bdx|)t^3\\
&\quad + (|afx| + |bex| + |cdx|)t^2 + (|bfx| + |cex|)t\\
&\qquad\qquad + |cfx| = 0
\end{aligned} \tag{16}$$

may be obtained by solving for the ratios in the two ways indicated by

$$\begin{aligned}
&a_i t^3 \cdot kt^2 + b_i t^3 \cdot kt + (c_i t^3 + d_i t^2 + e_i t + f_i)k + \sigma x_i = 0,\\
&(a_i t^3 + b_i t^2)kt^2 + c_i t^2 \cdot kt + (d_i t^2 + e_i t + f_i)k + \sigma x_i = 0.
\end{aligned} \tag{17}$$

Using the transformation mentioned above (which in this case consists in interchanging a_i and f_i, b_i and e_i, c_i and d_i with the subsequent change of t to $1/t$), the two remaining equations of the five may be obtained from the first two of (16). The third equation of (16) is invariant under the transformation. Obviously equations (15) can be solved for the ratios in as many ways as the coefficients of kt^2, kt, and k can be selected. It will be found convenient in actual work to use the scheme partly indicated in equations (17). That is, in the first and second solutions of equations (15) the coefficients of kt^2 and kt are $a_i t^{n-2}$, $b_i t^{n-2}$ and $a_i t^{n-2} + b_i t^{n-3}$, $c_i t^{n-3}$ respectively; according to the same scheme in the third solution the coefficients of kt^2 and kt are $a_i t^{n-2} + b_i t^{n-3} + c_i t^{n-4}$, $d_i t^{n-4}$, etc.

In (5) we have the equations of two lines different for different values of t, and the locus of their intersection is the R^2. In (14) we have three concurrent variable lines, and the locus of their intersection is the R^3. Hence in general the method consists in finding the locus of the points of concurrence of n concurrent lines subject to the condition that this point of concurrence be on the R^n.

PENNSYLVANIA STATE COLLEGE.
February, 1917.

FORD'S STUDIES ON DIVERGENT SERIES AND SUMMABILITY.

Studies on Divergent Series and Summability. By WALTER BURTON FORD. Michigan Science Series, Volume II. New York, The Macmillan Company, 1916. xi + 194 pp.

DURING the past twenty years there has been an ever increasing interest in the study of divergent series and their applications. Naturally a coexistent phenomenon has been a very large expansion in the volume of literature on this subject. An idea of the amount of this expansion may be gathered from the bibliography of Professor Ford's book, which, while not exhaustive, contains a list of some two hundred books and memoirs (principally memoirs), of which all but about twenty have appeared from 1895 on.

Thus it has become more and more difficult for one who has not followed recent work on divergent series to ascertain readily the known results in a certain branch of that field or the methods that have proved fruitful in studying certain aspects of the subject. This is alike a handicap for the experienced research worker whose investigations in other fields have naturally led to a consideration of divergent series, and to the beginner in research who feels attracted toward the subject of divergent series and wishes to orient himself rapidly in the field in order to find the avenues that may lead to new results.

To both of these classes of readers, as well as to many others, Professor Ford's admirable work will undoubtedly prove a boon. It presents in clear and concise fashion the fundamental features of each of the two grand divisions of divergent series, namely, asymptotic series and summable series, and in

certain subdivisions of each leads the reader to the present boundaries of knowledge, at the same time pointing the road to further advances. The appearance of such books, written by American mathematicians and published under the auspices of American universities, augurs well for the future development of mathematics in this country. All those who have this development at heart must feel grateful to Professor Ford and to the University of Michigan for the present contribution.

We turn now to a more detailed consideration of the book. As indicated in the previous paragraph it divides itself into two parts, one devoted to asymptotic series, the other to summable series. The first part comprises Chapters I–III, the second part Chapters IV and V.

In Chapter I certain fundamental theorems with regard to the Euler-Maclaurin sum formula* are derived, and their application to the study of asymptotic developments of functions is illustrated by a consideration of Stirling's series. This latter discussion is followed by a brief account of Poincaré's theory of asymptotic developments, which closes the chapter. The different theorems about the Euler-Maclaurin sum formula are concerned mainly with different forms for the remainder term in that formula, and are chosen from the point of view of their usefulness in obtaining asymptotic developments. It is certainly a convenience to have these important results collected in such compact and usable form.

Chapter II is devoted to the application of the general theorems of the previous chapter to the derivation of the asymptotic developments for a number of typical cases of fundamental importance. Two general classes of functions are considered, those defined by infinite products and those defined by infinite series. Beginning with rather simple examples, the writer proceeds by gradual steps to a point where he is able to treat the problem of obtaining the asymptotic development of the general integral function of order greater than zero. He also obtains a general theorem with regard to the asymptotic development of functions defined by power series whose coefficients satisfy a certain restrictive condition. The chapter as a whole forms an excellent intro-

* Professor Ford follows Barnes in using the designation Maclaurin sum formula instead of the above. However, as Euler's priority of discovery is well known, this usage seems to be lacking in historical accuracy.

duction to the literature that deals with the problem of determining the asymptotic development of a given function. The illustrative examples are well chosen with a view to giving the reader an adequate idea of the methods that have been found useful in attacking that problem.

In Chapter III a brief account is given of the progress that has been made in obtaining the asymptotic solutions of linear differential equations and linear difference equations. No demonstrations are given here; the writer contents himself with stating the principal results, giving numerous references to the literature and indicating certain important problems in the field whose solution is yet to be obtained.

In Chapter IV the reader is introduced to the second class of divergent series, namely, summable series. The writer selects six of the earlier and more widely used definitions of the sum of a divergent series and discusses certain of their properties. The definitions selected are the well known Cesàro and Hölder arithmetic mean definitions, Borel's exponential method, his integral definition and a generalization of the latter, and finally Leroy's extension of Borel's integral definition. The discussion of these definitions begins naturally with a proof of their consistency with the definition of convergence, i. e., a proof that any convergent series will be summable by the given definition to the value to which it converges. The treatment of consistency is made more compact and lucid by first proving a general lemma, by means of which the consistency of the various definitions is readily established.

The author then proceeds to define what he calls the boundary value condition. He wishes to consider as summable only series

$$\sum_{n=0}^{\infty} u_n \tag{1}$$

for which the corresponding power series

$$f(x) = \sum_{n=0}^{\infty} u_n x^n \tag{2}$$

has a radius of convergence equal to unity, and then accept only definitions of the sum of (1) for which

$$s = \lim_{x \to 1-0} f(x).$$

He has already stated in the preface his belief that some limitation of this sort is desirable and timely in order to avoid confusion and inconsistency in the study of series.

On this point the reviewer is not in agreement with Professor Ford. It is of course highly probable that many definitions that satisfy the condition of consistency will not be very useful in the study of divergent series, and it may be found desirable at some future time to exclude certain definitions that do satisfy this condition. However, the reviewer does not think the time is ripe for such exclusion and he further believes that the limitations that the author prescribes for the definition of summability are too narrow. He does not share Professor Ford's opinion, expressed in a footnote in the preface, that the present situation in the theory of divergent series is closely analogous to the one which caused Cauchy and Abel to rule out divergent series from the domain of analysis. At that time there was a decided vagueness with regard to the sense in which a divergent series could be considered as having a sum, and a lack of precision in the various attempts to define a sum. Whatever definitions have been proposed recently have been stated with all the precision of modern analysis, and, as far as the reviewer is aware, have not been chosen in a pure spirit of arbitrariness. They have been selected, either from their probable usefulness in studying certain interesting types of divergent series, or as natural outgrowths of earlier definitions or of attempts to frame general theories of divergent series.

If any limitations on the definition of summability are eventually agreed upon by workers in the field of divergent series, it seems natural to the reviewer that they should not exclude any series to which we would wish to ascribe a sum, provided we have a method of summation that gives to such a series a sum that is generally useful. The limitations suggested by Professor Ford do exclude many such series, for under these limitations no power series could be regarded as summable outside of its circle of convergence. Yet such a series can be summed within the polygon of summability by Borel's integral definition to a value which is the analytic extension of the function defined within the circle of convergence. It is surely convenient to ascribe this value to the series, and since we can obtain the value by one of the standard methods of summation, it would seem logical to include such a series in the class defined as summable.

After defining the boundary value condition, the chapter under consideration continues with a discussion of the relationship to this condition of the six definitions mentioned above. Following this, conditions are obtained under which these definitions possess certain other properties. The author considers the effect on the summability and value of a series of adding terms to or subtracting terms from the beginning of the series, the nature of the series formed by adding corresponding terms of two summable series and that formed by taking the Cauchy product. The results are summarized on page 91. In the remaining sections of the chapter the writer discusses absolutely summable series (Borel), uniform summability, and a few important properties of series summable by the Cesàro (or Hölder) definition, or by Borel's integral definition.

Chapter V deals with the application of the theory of summability to certain developments in orthogonal functions, namely to Fourier's series and some of the allied developments. In this connection the convergence of the developments, as a special case of summability, is considered as well. While the author does not in general obtain results not previously known, he makes here a distinct contribution to the subject in that he develops a general theory of summability of developments of this type, out of which the results for particular developments may be obtained as special cases. This serves to unify the whole treatment and to bring into greater prominence the essential features of the problem. The method of treatment is an extension of Dini's discussion of the convergence of developments of this type.

The chapter is divided into four parts. In the first part the author discusses the simplest case, the convergence and summability of the ordinary Fourier's series. In connection with this discussion he makes clear to the reader what are the essential steps in building up a general theory that will apply to developments of this type. In the next part he develops a number of general theorems about the representation of arbitrary functions by definite integrals, which form the basis for this general theory. In the third part he shows how Cauchy's calculus of residues can be used to advantage in applying the theorems of the previous part to the developments in question. He also discusses certain general properties of the functions appearing in the terms of the developments, which are useful in studying their convergence or summability.

In the fourth and final part of the chapter we find a discussion of three different types of developments, namely a class of important sine developments,* developments in Bessel's functions and developments in Legendre's functions. All these discussions are carried through on the basis of the author's general theory. The results, while not always the most general that have been obtained, are broad enough to show the power of the general theory. There is just one point in which results sufficiently complete for the applications have not been secured. In discussing the developments in terms of Bessel's functions, no information is obtained with regard to the convergence or summability of the series at the origin, or its uniform convergence or summability in the neighborhood of the origin. Without information of this sort we are not able to carry through some of the applications of these developments to problems in mathematical physics. It is true that most discussions of the developments in Bessel's functions leave this same point unsettled, but it is obviously a matter of much importance, and any treatment that overlooks it is incomplete in an essential point.

In connection with his discussion of the developments in Bessel's functions Professor Ford has very properly called particular attention to the fact that the first adequate discussion of the convergence of the Bessel's developments for points in the interval $0 < x \leqq 1$ was given by Dini, since the latter's priority in this matter has not received due recognition from certain writers.†

Chapter IV concludes with a discussion of the convergence and summability of the developments in Legendre's functions. The results here, though not the most general that have been

* These developments have in the successive terms of the series trigonometric functions of the form $\sin \lambda_n x$, where the λ's are the roots of the equation

$$z \cos z + p \sin z = 0.$$

† Cf. for example p. 374 of Whittaker and Watson's Modern Analysis, where the development theorem is stated and ascribed to Hobson. It is true that Hobson's result went beyond Dini's in that he considered functions having a Lebesgue integral and discussed uniform convergence. However, making allowance for a few errors in formulas, the result stated by Whittaker and Watson was fully established by Dini, and it is unfortunate that Dini's work should have been referred to by these writers in such a way as to emphasize the errors and overlook the substantial contribution.

obtained,* are complete in the sense that both convergence and summability are established under fairly broad conditions for the entire interval $(-1 \leqq x \leqq 1)$, thus including the points $x = -1$ and $x = 1$, which are exceptional in the same sense that the point $x = 0$ is for the developments in Bessel's functions.

At the beginning of this review two classes of readers to whom Professor Ford's book should prove very useful, have been mentioned. In concluding, a third class should be noted, namely those who are already actively engaged in research work in the field of divergent series. The literature in this field has grown so rapidly in such a short time that it is sometimes difficult, even for those who have made a point of following it, to locate readily a particular result. Professor Ford's references to the literature are on the whole very complete and have evidently been made the object of much care. In this way his book furnishes readier access, not only to the many valuable results it contains, but also to a still greater number of related results in the literature of divergent series.

A list of errata that have been noted is appended:

P. vii, eq. (1)	for	$\frac{B_3}{5.6}\frac{1}{x^6}$	read	$\frac{B_3}{5.6}\frac{1}{x^5}$
P. 5, l. 7	"	(5)	"	(6)
P. 8, l. 2	"	§ 5	"	§ 6
P. 10, l. 9	"	$\epsilon > h$	"	$\epsilon < h$
P. 12, l. 1	"	$\sum_{x=a}^{b-h}$	"	$\sum_{x=a}^{b-h}$
P. 22, footnote 27	"	$e^{-(1/x)}$	"	e^{-x}
P. 38, l. 22	"	$x = +\theta_y$	"	$x = y + \theta_y$
P. 50, eq. (48) and following eq.	"	$(-z)^{a-(1/2)-iy}$	"	$(-z)^{a-(1/2)+iy}$
P. 52, l. 9	"	$-\pi < \varphi < \pi$	"	$-2\pi < \varphi < 0$
P. 66, l. 8	insert the words *the given* before *equation*			
P. 93, fig. 6	for	λ	read	2λ
P. 149, eq. 116	"	$-\frac{1}{\sqrt{2\pi z}}$	"	$-\frac{i}{\sqrt{2\pi z}}$
P. 149, the eq. preceding (119)	"	$(2\upsilon-1)$ in exponents	"	$(2\upsilon+1)$
P. 155, ll. 15, 22 and 33	"	$\begin{cases} t > 0 \\ t < 0 \end{cases}$	"	$\begin{cases} t > \epsilon \\ t < \epsilon \end{cases}$
P. 162, last line of footnote	"	Nielson	"	Nielsen
P. 184, l. 5	"	Allerdice	"	Allardice

CHARLES N. MOORE.

* In the opinion of the reviewer attention should have been called to Gronwall's very exhaustive discussion of the summability of the developments in Legendre's functions (cf. *Math. Annalen*, vol. 75 (1914), p. 321) in connection with the references to the literature in this chapter.

SHORTER NOTICES.

The Human Worth of Rigorous Thinking. By CASSIUS J. KEYSER. Columbia University Press, 1916. 8vo, vii+314 pp. $1.75.

THESE essays and addresses of Professor Keyser are collected together in one volume as the result of fifteen years' consideration of various phases of mathematics, particularly of its significance as a branch of human knowledge. Everyone who has heard or read the different articles already knows that the author sees things with a poet's eye, and hears the strains of subtle music that emanate from this highest ether of the reason. The closing sentence of his book, and of the lecture widely delivered on mathematics, finds the deep-centered source of not only mathematics but also of science in the "joyous mood of the eternal Being" and both are considered to be a "sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eaglet sporting with the mountain winds." For him "mathematics is but the ideal to which all thinking, by an inevitable process and law of the human spirit, constantly aspires." Consequently to challenge the value of mathematics as a study worthy of humanity is to challenge the worth of all thinking. Its significance to man is beyond measure, for "Transcending the flux of the sensuous universe, there exists a stable world of pure thought, a divinely ordered world of ideas, accessible to man, free from the mad dance of time, infinite and eternal." Its teaching may be so humanized that it will become a delight and a pleasure to all students, for "mathematics is precisely the ideal handling of the problems of life, and the central ideas of the science, the great concepts about which its stately doctrines have been built up, are precisely the chief ideas with which life must always deal, and which, as it tumbles and rolls about them through time and space, give it its interests and problems, and its order and rationality." The first three and the last two addresses of the collection are devoted primarily to considerations of the sort we have just mentioned.

There are four essays dealing with such fundamentals of thought as the figure and dimensions of the universe, hyperspace, infinity, and the existence of the hypercosmic. Two

others are closely related to these in that they discuss the logical reduction of mathematics and the multiple interpretations of systems of postulates and the deductions from them. In these we find the same lofty strain insisting upon the ability of the reason to find ultimate knowledge in mathematical theorems. "It is not in the world of sense, however precious it is and ineffably wonderful and beautiful, nor yet in the still finer and ampler world of imagination, but it is in the world of conception and thought that the human intellect attains its appropriate freedom—a freedom without any limitation save the necessity of being consistent." He is audacious enough to foresee an increase in our imagining powers so that even the ability to picture n-dimensional spaces may some day be gained by the race. In the hypercosmic he finds a world of pure thought "where is every type of order and manner of correlation and variety of relationship," rich in æsthetic interests, furnishing a sublimed and supersensuous art, the sure support of religious aspiration, permanent through all time. In hyperspaces he finds entities that have precisely whatever kind of existence may be attributed to the space of ordinary geometry. Whether such entities are found by the intellect or are created by it makes little difference.

There are also four essays on education, discussing the basis of a liberal education, graduate instruction, the functions of a university, and research in American universities. He takes issue with the doctrine that all subjects are equally valuable in a curriculum, pointing out that there are deep-rooted instincts in mankind which seek for certain things and for which any curriculum must provide. The study of the history of human activity and thought is a necessity, as well as the study of natural science; the study of mathematics, logic, or rigorous thinking in some form is not to be omitted, nor is that of the social character of man; but in a way surmounting them all is the study of beauty in all its forms, for without this, life would indeed be poor and depressing. Although the various topics that will respond to these needs may vary from century to century yet in some form or other they must always be present if education is really to be liberal in the highest sense. Would that everyone could appreciate the deep underlying principles set forth here, in this day of educational fads and crude radicalism! A university he finds is "the offspring and the appointed agent of the spirit of

inquiry, expression and servant of that imperious curiosity which in a measure impels all men and women, but with an urgency like destiny literally *drives* men and women of genius, to seek to know and to teach to their fellows whatsoever is worthy in all that has been discovered or thought, spoken and done in the world, and at the same time seeks to extend the empire of understanding endlessly in all directions throughout the infinite domain of the uncharted and unknown." As to research, the author believes in the three-fold organization of a university staff, the administration, the teaching staff, and the research staff. He characterizes his conceptions of the three thus: the great administrator is a man of constructive genius, the great teacher is a source of inspiration, the great investigator is a discloser of the harmonies and invariance hid beneath the surface of seeming disorder and of ceaseless change.

The delightful style of the author, permeated as it is with lofty sentiment, keen appreciation of beauty, wealth of imagery, striking illustrations, wide choice of terms, classical allusion, and warm feeling, makes the reading of these essays on rather abstract philosophical topics a pleasure as well as a profit. Every student of mathematics should read the book and catch the inspiration of enthusiasm for the divine science. "Mathematics is, in many ways, the most precious response that the human spirit has made to the call of the infinite and eternal. It is man's best revelation of the 'Deep Base of the World.'"

JAMES BYRNIE SHAW.

Four Lectures on Mathematics. (Delivered at Columbia University in 1911.) By J. HADAMARD, Member of the Institute, Professor in the Collège de France and in the Ecole Polytechnique, Lecturer in Mathematics and Mathematical Physics in Columbia University for 1911. New York, Columbia University Press, 1915. v + 52 pp.

ALL those who have had the pleasure of hearing Professor Hadamard's lectures have doubtless remarked his unusual facility in opening up wide vistas in the course of a relatively brief discussion. It is natural to expect such a facility to appear at greatest advantage in a set of lectures that are intended to be primarily inspirational, such as the above, and in this case the expectation is amply realized. By the omission of practically all technical details, Professor Hadamard has

succeeded in giving a bird's-eye view of some very extensive domains of mathematics in the course of these four lectures.

The first lecture deals with solutions of linear partial differential equations determined by boundary conditions, and the central topics are Cauchy's problem and Dirichlet's problem. The essential differences in nature of these two problems are set forth in clear and elegant fashion, and an illuminating discussion of their relationship to certain important physical problems is given. The lecture closes with a quotation from Poincaré on the service of physics to analysis in furnishing important problems whose solutions can frequently be predicted on physical grounds.

The second lecture deals with contemporary researches in differential, integral, and integro-differential equations. In discussing the solution of Dirichlet's problem by means of integral equations, the writer reminds us that the complete solution of a mathematical problem suggested by a physical problem may only be arrived at after interest in the physical problem has been lost. But at the same time he consoles us by pointing out the compensating feature that the mathematical solution may be of very great use in connection with other physical problems of greater actuality. The rest of the lecture concerns itself mainly with important investigations in differential and integro-differential equations that have been inspired by the consideration of certain physical problems. It is pointed out that under the present hypothesis of the discrete structure of matter, physical problems tend to lead to ordinary differential equations rather than to partial ones.

The third lecture is devoted to a discussion of analysis situs. The essential nature of this important branch of mathematics and its invaluable assistance in developing the theory of certain other branches, is very clearly brought out in brief compass. It is shown that in dealing with certain types of problems, analysis alone is insufficient to resolve all the difficulties without the powerful aid rendered by analysis situs. The rôle of each branch is shown in very illuminating fashion by comparison with a map of a large country given on a series of partial leaves that can only be put together by means of an "assembling table" showing the general disposition of the leaves.

The fourth and final lecture is devoted to elementary solu-

tions of partial differential equations and Green's functions. A brief but comprehensive résumé of recent investigations concerned with these two topics is given. The writer points out that the important difference between the Green's function and the elementary solution of the corresponding differential equation, corresponds to a similar difference between Cauchy's problem and Dirichlet's problem. That is to say the Green's function, like Dirichlet's problem, depends very closely on the form of a certain surface or hypersurface, whereas the elementary solution, like Cauchy's problem, does not. From this fact it is readily seen that considerations of analysis situs will play an important rôle in the study of Green's functions. Hence these functions are related to all the principal topics of the preceding lectures, and therefore, as the writer expresses it, a discussion of them forms an appropriate conclusion.

CHARLES N. MOORE.

Die elliptischen Funktionen und ihre Anwendungen. Erster Teil. By ROBERT FRICKE. Leipzig, B. G. Teubner, 1915. x+500 pp. Price 22 Marks.

THE present volume is the first of a series of three which Dr. Fricke proposes to write on the elliptic functions and their applications. It appeared in October, 1915, and is devoted to the function theoretic and analytic bases of the theory of elliptic functions. One would naturally expect that a treatise on elliptic functions from the pen of Dr. Fricke would follow the lines of thought developed by Klein and his students thirty-odd years ago. Consequently, on turning the pages of the present volume, one is not surprised to be reminded again and again of modes of thought, of formulas, and of geometric diagrams made familiar through the Klein-Fricke Modulfunktionen. Dr. Fricke refers to this when he writes in the preface: "That I should adhere in the main to the methods of presentation, the use of the invariant theory, geometric representation, and so forth, which more than thirty years of close scientific companionship with my teacher and friend F. Klein have made my own, I may regard as self-evident."

The introduction, consisting of 105 pages, is devoted to an exposition of theorems concerning analytic functions of a single complex variable. This material is made to lead up to a statement of the basic problems of the theory of elliptic func-

tions; namely, a study of the algebraic functions and their integrals on a Riemann surface whose deficiency is $p = 1$, and a classification of these functions into essentially different stages (Stufentheorie). The introduction closes with an application of the Fuchs' theory to linear differential equations of the second order, in particular to the hypergeometric differential equation and hypergeometric series.

The remainder of the volume is divided into two parts (Abschnitte), of which the first, consisting of five chapters, is devoted to the theory of elliptic functions of the first stage (Stufe). This, as we know, is the "Weierstrass theory" with its characteristic rational invariants g_2, g_3 and the corresponding normal form of the integral of first kind

$$u = \int \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$

The discussion contains the usual materials with the invariant theory and the group theory always in the foregound. The first part closes with a discussion of the "differential equation of the periods" and the related Schwarz differential expression.

The second part, consisting of three chapters, opens with a discussion of irrational invariants of the algebraic form $f(z)$ of the fourth degree and we soon reach the characteristic irrational invariant λ (anharmonic ratio) of the second stage, and the irrational invariant μ (octahedral irrationality), characteristic of the fourth stage, together with the corresponding normal forms of the integral of first kind

$$u = \int \frac{dz}{\sqrt{z(1-z)(1-\lambda z)}},$$

$$u = \int \frac{dz}{\sqrt{(1-z^2)(1-\mu^4 z^2)}}.$$

The first of these is the Riemann normal integral, and the second resembles very closely the Legendre-Jacobi normal form, but our author prefers to call it the Abel normal form, since Abel used it in his Recherches sur les Fonctions elliptiques. The Legendre-Jacobi normal form differs from the Abel normal form in having k^2 in place of μ^4. Moreover the Jacobi elliptic functions belong to the second stage in the Stufentheorie and not to the fourth.

The explicit statement of Klein's Stufentheorie occurs in the second chapter. This theory depends upon the group of linear substitutions $\Gamma^{(u,\,\omega)}$, that is,

$$
\begin{aligned}
u' &= u + m_1\omega_1 + m_2\omega_2, \\
\omega_1' &= \qquad\quad \alpha\omega_1 + \beta\omega_2, \\
\omega_2' &= \qquad\quad \gamma\omega_1 + \delta\omega_2,
\end{aligned}
\tag{1}
$$

where m_1, m_2, α, β, γ, δ are integers and $\alpha\delta - \beta\gamma = 1$, and its congruence subgroups; the principal congruence subgroup of nth stage being defined by the congruences

$$m_1 \equiv 0, \quad m_2 \equiv 0, \quad \alpha \equiv \delta \equiv 1, \quad \beta \equiv \gamma \equiv 0 \pmod{n}. \tag{2}$$

The group $\Gamma^{(u)}$; that is,

$$u' = u + m_1\omega_1 + m_2\omega_2,$$

and the modular group $\Gamma^{(\omega)}$ are considered separately. The principal congruence subgroup of nth stage of $\Gamma^{(u)}$ is defined by the first two congruences of (2), and the principal congruence subgroup of nth stage of $\Gamma^{(\omega)}$ is defined by the last four congruences of (2). The theory of the modular group and its principal congruence subgroups is contained in the Modulfunktionen (1890). The definition of the discontinuity domains for the principal congruence subgroups of nth stage of $\Gamma^{(u)}$ and of $\Gamma^{(\omega)}$ follow, and we are led, finally, to the following definition of an elliptic function of nth stage.

An elliptic function of nth stage is a uniform homogeneous function $\psi(u/\omega_1, \omega_2)$ of integral dimensions d of the three arguments u, ω_1, ω_2 which is invariant (covariant) under the substitutions of the principal congruence subgroup of nth stage of $\Gamma^{(u,\,\omega)}$; for fixed values of ω_1, ω_2, it is an analytic function of u, free from essential singularities in the discontinuity domain of the principal congruence subgroup of nth stage of $\Gamma^{(u)}$ and is defined for the entire u-plane except for $u = \infty$; and for a fixed value of u/ω_2

$$\omega_2^{-d}\psi(u/\omega_1, \omega_2) = \psi\left(\frac{u}{\omega_2}\middle|\omega, 1\right) \quad \left[\omega = \frac{\omega_1}{\omega_2}\right]$$

is an analytic function of ω, free from essential singularities within the discontinuity domain of the principal congruence subgroup of nth stage of the modular group $\Gamma^{(\omega)}$, and is defined for the positive ω-half plane.

Nearly a quarter of a century ago Professor Klein expressed the hope that he might sometime be able to treat the whole theory of elliptic functions from this point of view,* and the present text, when completed, may be regarded as in a sense a fulfillment of that hope.

The Jacobi functions, $\operatorname{sn} w$, $\operatorname{cn} w$, $\operatorname{dn} w$, now appear as elliptic functions of the second stage and the remainder of the volume is devoted to their properties, their relations to the Weierstrass functions, and to the properties of the allied theta functions.

Dr. Fricke's style is well known the world over and needs no comment. The text is free from errors, due credit being given by the author to Fräulein Dr. H. Petersen for careful proof-reading. The book appeared from the Teubner press "in spite of all the difficulties of the time, without unusual interruptions." In conclusion we may express the hope that nothing will interfere with the publication of the two remaining volumes.

L. WAYLAND DOWLING.

Problems in the Calculus, with Formulas and Suggestions. By DAVID D. LEIB, Ph.D. Boston, Ginn and Company, 1915. xii+224 pp. Price $1.00.

THE first impression conveyed by the title of this book, that it may be a collection of problems of the sort discussed by Professor Archibald in the BULLETIN of June, 1914, is corrected by the first few words of the preface. The book is said to be "the outgrowth of lists of problems prepared by the author to supplement the textbook," and in it we find "a supplementary list of workable problems on any topic ordinarily included in a general course in the calculus."

Preceding each set of exercises is a brief statement of the formulas and methods applicable thereto, with warnings against some common errors. In fact it seems to the reviewer that too careful a classification and too definite directions are given to permit, much less encourage, the development of initiative, versatility, and flexibility on the part of the student. When he has worked through a reasonable number of examples in his text-book, where properly enough most of them are carefully classified, the best additional preparation for the

* Evanston Colloquium (Aug. 28–Sept. 9, 1893). Lecture X.

unassorted difficulties he will meet in physics, mechanics, or elsewhere would seem to be furnished by examples in which the method of attack is left for him to discover. This is especially true in the case of formal integration; many students can "turn the crank" when they know what crank to turn, but are helpless when it comes to selecting a suitable method for themselves. This book of problems would be more helpful to them if the author had written each problem on formal integration for instance on a separate piece of paper and had secured the cooperation of a playful kitten in completing the arrangement.

The intentional omission of the answers to a large number of examples should assist the development of students, but one would hardly make that claim for the inaccuracy which is threatened by the author's statement that "no one has formally verified the answers." The reviewer has not done so in many cases, but would call attention to errors in parts (*b*), (*e*), (*f*), (*g*), (*i*) of example 3, exercise XCIX and in examples 3 and 15 in exercise CII. The last-named example is the approximate integration of

$$\int_0^{\frac{1}{2}\pi} \frac{dx}{1 + \cos x}.$$

In the first place, it hardly seems worth while to use approximate rules for more than one or two examples that can be done easily by direct integration; or at least it is not worth while to preserve many such problems and answers in a book. In the second place, it is hardly worth while to employ the methods of approximate integration unless it is done accurately enough to give better results than inspection, counting squares, or guessing in any other form. The answer given for this problem, 1.057 as compared with the correct 1, is surely beyond permissible limits of error. It apparently results from a hasty use of the trapezoidal rule, which inspection shows to be inexact in this case. The same unwise use of this rule is suggested in example 3 of the same exercise, to find by four ordinates the value of $\int_0^1 \sqrt{1 + x^4}\, dx$. If this is treated as an area, the slope of the curve at the upper point is $\sqrt{2}$, and the slope of the upper boundary of the last trapezoid is less than 1; obviously the trapezoidal rule will be inaccurate. The answer

given, 1.07, seems to be the result of hasty work; accurate use of the trapezoidal rule yields 1.103, as compared with the exact value, 1.0894, obtained by the aid of Γ functions and verified by Simpson's rule with five ordinates. The error due to the trapezoidal rule is nominally only 1.3 per cent., but inspection shows the area to be a unit square surmounted by a small area, and for this latter even an accurate use of the rule suggested results in an error of about 15 per cent. Such approximate integration seems unsatisfactory to the reviewer, although it must be admitted that other books appear to sanction it.

The points open to criticism—the too orderly arrangement, inaccuracies in certain answers, and undesirable features in a few problems—are flaws which limit rather than destroy the usefulness of the book. It is certain that many teachers will find it convenient to put in the hands of students for supplementary work, and more will find it a very satisfactory source from which to draw, for classroom work or tests, problems which will be new to their students.

R. W. BURGESS.

Vom periodischen Dezimalbruch zur Zahlentheorie. By ALFRED LEMAN. Leipzig, Teubner, 1916. iv+59 pp. Price 80 Pfennige.

THE aim of this little book is to present the main properties of periodic decimals as material for a concrete introduction to the more elementary topics of the theory of numbers. The concept congruence is introduced on page 16 and Fermat's theorem is presented, illustrated and proved on pages 21–23, —each topic being a natural sequel to concrete questions and results concerning periodic decimals.

Periodic fractions to bases other than 10 are treated briefly in the final chapter 8, although on page 51 this part of the theory is said to be treated in chapters 8 and 9.

There is a list of about 15 papers, including most of the earliest ones. There is no mention of the early MS. by Leibniz; Henry Goodwyn's tables, 1816–23; the papers by Poselger, 1827; Bredow, 1834; Midy, 1836; Catalan, Thibault, and Sornin in *Nouvelles Ann. Math.*, 1842, 1843, 1846; Desmarest's text, 1852; Hudson's excellent paper in Oxford, *Cambridge and Dublin Messenger of Mathematics*, 1864, pages 1–6—not to cite various earlier papers and a hundred later ones. The topic is not exhaustively treated as claimed.

In the notes on page 54, the statements concerning perfect numbers and the largest known prime are not up to date.

However, the author has succeeded in his aim to write an attractive account of the main properties of periodic decimals and to use that topic as a concrete means to acquaint the immature reader with some important theorems of the theory of numbers and to arouse his curiosity to pursue the theory further.

L. E. DICKSON.

Compendio de Álgebra de Abenbéder. Texto árabe, traducción y estudio por JOSÉ A. SÁNCHEZ PÉREZ. Madrid, 1916. (Junta para Ampliación de Estudios e Investigaciones científicas, Centro de Estudios históricos.)

THE translator, who has an enthusiastic hope that some day an adequate history of mathematics in Spain will be written, sees in this manuscript a contribution of material for such a book. But the translation is of value for another reason: it makes accessible to the mathematical world in general another of the works compiled by the Mussulmans between the eighth and fifteenth centuries. Sr. Pérez begins his introduction with a brief account of certain phases of the history of mathematics in his own country and closes it by thirty pages of discussion of questions relating to the contents of this document and its authorship and date.

The Compendio de Álgebra is contained in manuscript 936 of the Escurial library (Arabic section). There are forty-six folios in Arabic characters of Spanish type. The date, as given in the document, is the year 744 since the Hegira. Abenbéder understands that the object of algebra is the solution of equations. He explains his subject in the form of ordinary discourse, without employing the notations of algebra. The work is divided into two parts—the theoretical and the practical. In the former, the first six "questions" treat the six forms of equations given by Al-Khowarizmi. Some of the particular equations used are those met with so frequently in these early texts: for example, (in our notation) $x^2 + 10x = 39$ and $x^2 + 21 = 10x$. There are also six "chapters" explaining the fundamental operations with the square roots of numbers and six others dealing with the rules of signs and with the squares and cubes of the unknowns.

The practical part of the book starts out with six problems illustrating the six types of equations. Five of these might well be placed in the next chapter, which contains eleven problems "concerning ten." In each of these sixteen exercises, the number ten is to be divided into two parts according to to some specified condition: for example, the product of the parts divided by their difference equals five and one fourth. Then follow seven collections containing forty-eight examples in all. Half of these are "problems of the squares"; such as, one more than one third of a square multiplied by one more than one fourth of the square equals twenty. Another group of five questions about soldiers involves arithmetical progressions. There are also "problems of commerce," "problems of the gifts," and others concerning our old friends the couriers, who have, by various methods of travel, been pursuing one another so industriously through the pages of algebra textbooks for centuries. After a formal statement that the end of the book of "chéber y almocábala" has been reached, there are appended a problem involving an arithmetic progression and also three meager rules for the solution of the three types of the complete quadratic.

E. B. COWLEY.

First Year Mathematics. By GEORGE W. EVANS and JOHN A. MARSH. New York, Chas. E. Merrill Company, 1916.

As its name indicates, this work is intended for use in the first year of the high-school course, and is a correlation of elementary algebra, plane geometry and the fundamental ideas of coordinate and locus.

In Chapter 1, simple equations in one unknown are introduced as a means of abbreviating arithmetical processes, with applications to ratio, linear and angular measurement, valuation problems, angles and angle relations and circular measurement. It may be noted that the term "stripe" is introduced in this chapter to denote a pair of parallel lines. Also, that it is pointed out in detail that precision of measurement is indicated by the number of significant figures in the result rather than by the number of decimal places. In explaining negative quantities the historical method is followed by assigning to the negative the primitive idea of a shortage to be made up, or caused to disappear, by the addition of a quantity sufficient to cover the shortage. The

word shortage is the term actually used in the text, with a note to the effect that the process of making up a shortage is the real significance of the Arab word algebra, of which the English equivalent is "restoration."

Chapter 2 takes up approximate computations, including multiplication, division, and square root, with practical rules for fixing the decimal point and determining the accuracy of the result.

Chapter 3 relates to the measurement of areas of plane figures, including the triangle, trapezoid, and circle, and the evaluation of simple formulas.

Chapter 4 is devoted to the solution of fractional equations and certain standard types of problems, notably that of numbers formed by certain digits, the problem of relative velocities, and the ancient problem of ability and time. The problems in this chapter seem to be either irrelevant or of merely historical interest, and might well be replaced by problems of modern application and significance.

Chapter 5 considers simple transformations of algebraic expressions such as transposition and the use of parentheses, and the multiplication and division of polynomials. The algebraic negative is here again explained as a shortage, such for example as arises when a merchant is unable to fill an order because of insufficient stock on hand.

Chapter 6 explains the distinction between an equation of condition and an algebraic identity, and also takes up the factoring of quadratic expressions.

Chapter 7 is devoted to the solution of quadratic equations. An important feature of this discussion is the careful and complete explanation given of the practical meaning of like and unlike roots, of negative and imaginary roots, of meaningless answers, etc.

In Chapter 8 the elementary theorems of plane geometry are treated, including the properties of similar triangles, the ratio properties of triangles, and the Pythagorean theorem, the latter also being treated by the aid of the trigonometric tangent.

Chapter 9 explains the fundamental idea of coordinate and locus, and illustrates by plotting data giving rise to a linear graph. To illustrate the relation of converse statements, such as "If an equation is of the first degree its locus is a straight line," and its converse "If a locus is a straight line

its equation is of the first degree," the pupil is required to give the converse of certain other statements, such as "If my straw hat is ruined a horse has stepped on it."

In Chapter 10 simultaneous linear equations are solved by elimination and by combination.

In Chapter 11 simultaneous equations, including quadratics, are solved by the method of substitution, and also graphically by the intersection of their loci.

Chapter 12 consists of 30 pages of supplementary problems for review.

As a brief summary of the treatment, it may be said that formal definitions are reduced to a minimum, and every effort is made to appeal to the common sense of the pupil. The text is so arranged as to give the pupil a clear idea of the meaning and purpose of algebra and geometry, rather than to set forth a logical and conventional system of mathematical doctrine. This is primarily the ideal to be attained in the coordination of elementary mathematics, and as this idea seems to have been clearly the guiding principle in the selection and arrangement of material in this case, it is interesting to note the vitality and unity it gives to the text. Such a one volume text necessarily covers a limited field, but within its limits it is, with a few exceptions, thoroughly modern in its spirit and aims, as well as eminently teachable.

S. E. SLOCUM.

Second-Year Mathematics for Secondary Schools. 2d edition. By E. R. BRESLICH. University of Chicago Press, 1916.

THIS book is one of the numerous recent attempts to correlate elementary algebra, geometry, and trigonometry for purposes of instruction. The idea of correlation is of course one of great possibilities, but to be successfully realized it must be based on some central and unifying principle, such, for instance, as the function concept advocated by Klein. A careful examination of the present work fails to reveal any such principle of selection or arrangement, and leaves the impression of a haphazard collection of unrelated topics.

In order that the book may be judged on its own merits and not by the opinion of the reviewer, the following brief summary of its contents is given.

Chapter 1 is a tabular statement of the geometric theorems

and constructions included in the preceding volume on first year mathematics.

Chapter 2 is a brief explanation of logical statement, fallacies, and methods of proof, which could hardly be appreciated by pupils at this point, before having had sufficient practice to distinguish between the methods described and the difficulties involved.

Chapter 3 relates to algebra, including simultaneous linear equations, elimination by combination and by substitution, and by the intersection of the graphs of their loci. This arrangement is meritorious, but the treatment is entirely too brief and is accompanied by very few problems.

Chapter 4 takes up geometry, beginning with the properties of quadrilaterals and parallelograms. This is interrupted by four quadratic equations to be solved algebraically, followed by three problems on loci, a page and a half on prisms and a page and a half on dihedral angles. This chapter shows plainly the lack of coordination which characterizes the book.

Chapter 5 treats of proportional line segments, supplemented by properties of transversals intercepted by a system of parallel planes, and includes photographs of two railway bridges.

Chapter 6 begins with the theory of proportion, is interrupted by three pages on the factoring of polynomials, returns to proportion, and ends by showing the relation of direct and inverse variation to proportion.

Chapter 7 treats of similar plane figures, but digresses to explain the graphical method for finding mechanically the quotient of two arithmetical numbers. The objection to such an irrelevant digression is that it violates the principle of correlation which aims at concentrating the pupils' thought instead of diverting it.

Chapter 8 starts out by defining the projection of one line on another, then proceeds to the simplification of radicals, passing next to a discussion of the theorem of Pythagoras with historical notes and a picture of Fermat. The formula for the solution of quadratic equations is then given, and the chapter concludes by generalizing the theorem of Pythagoras. How this arrangement facilitates instruction is not fully apparent.

Chapter 9 is mainly on trigonometry. The sine, cosine and tangent are defined, and brief tables of these functions given, with applications to the solution of plane triangles,

followed by the derivation of two trigonometric identities, after which comes the algebraic and graphical solutions of a quadratic with a linear equation.

Chapter 10 relates to theorems on the circle, with historical notes and a photograph of Dryburgh Abbey.

Chapters 11 and 12 are on geometry, discussing the measurement of angles by circular arcs, and proportional line segments in circles.

Chapter 13 is on algebra and considers fractional equations including trigonometric identities.

Chapter 14 swings back to geometry, taking up geometric inequalities mainly applied to line segments.

Chapter 15 begins with dihedral angles and concludes with certain theorems on the sphere, enlivened by a photograph of the cathedral of S. Maria del Fiore.

Chapter 16 treats of geometric loci, including among other things definitions of the circumcenter, incenter, excenter, orthocenter, the nine point circle and the Eulerian line.

Chapter 17 is on the relation of a circle to its inscribed and circumscribed circles, and includes the determination of the value of π. This is illustrated by a photograph of Gauss, with historical notes, and photos of two ancient German buildings.

Chapter 18 begins with a few theorems on plane areas, passing from this to the solution of literal equations in one and two unknowns, then back again to the discussion of certain theorems on areas of triangles, and ending with the factoring of polynomials.

Chapter 19 concludes the work with a discussion of areas of polygons, illustrated with photographs of two peculiar buildings, which are not named but apparently represent what is called the Chicago style of architecture.

The modern demand for economy of time in education will eventually lead to correlation of mathematics in which the central idea will be to cultivate in the pupil the habit of mathematical thought and exact expression, to give him equal facility in the application of algebraic and geometric methods, and to make the subject matter vital by modern applications and interpretation. The outline given above seems to indicate that the opportunity to accomplish these results has been almost entirely overlooked in the present work, as the arrangement has no advantage over similar material selected at random from standard texts. S. E. SLOCUM.

A Course on the Solution of Spherical Triangles for the Mathematical Laboratory. By HERBERT BELL, M.A., B.Sc., Assistant in Natural Philosophy in the University of Edinburgh. London, G. Bell and Sons, Limited, 1915. viii + 66 pp.

THIS little book is No. 5 of the Edinburgh Mathematical Tracts of which Nos. 1, 2, and 4 are also manuals for the mathematical laboratory. It presents an account of various methods, numerical and graphical, of solution of spherical triangles.

The author keeps his object clearly in mind, goes directly at it, maintains a good proportion of parts in definition, exposition, and illustration, and the result is a practical syllabus of the subject in brief space. The value of this manual is considerably enhanced for the inexperienced computer by the foresight of the author in going into details to exhibit and illustrate methods and devices which arise from experience which the beginner has not had, and are not immediately evident. On the other hand the reader is not confused with long expositions where they are not needed, and when he has once been over the contents and has acquired some experience in computing he will find the book what it was intended to be, a useful and practical manual.

The first chapter deals with the use of logarithmic tables, including addition and subtraction logarithms, and questions of precision. Seven place tables are used in the examples. The second chapter is an exposition of the general spherical triangle, the cases which arise on the basis of D'Ocagne's classification, and the fundamental formulas, including those for solution by means of auxiliary angles. The third and fourth chapters are devoted to the numerical solution of the right-angled and the general spherical triangle, respectively.

Chapter V treats certain special applications to navigation and astronomy, including great circle sailing, finding the longitude at sea, reducing an angle to the horizon, and conversion of star coordinates. In Chapter VI graphical methods of solution are set forth at some length. Here are given Monge's method, the Emerson-Langley construction, nomograms, including the analemma, Chauvenet's solver, and straight-line nomograms.

A. M. KENYON.

United States Life Tables: 1910. Prepared in the Division of Vital Statistics of the Bureau of the Census under the supervision of JAMES W. GLOVER. Washington, D. C., Bureau of the Census, 1916. 65 pp.

THIS work includes twenty-five life tables derived from twenty-five classes of persons of the population of the original registration states. The statistical data for these tables are obtained from the estimated population within these states as of July 1, 1910, and from the corresponding deaths in the calendar years 1909, 1910, and 1911. Osculatory interpolation retaining fifth differences was employed in distributing into yearly age groups the numbers given in quinquennial age groups. The first table in the book (pages 16–17) is derived from the entire population of the original registration states. The remaining twenty-four tables are obtained by classifications of this population into male and female, white and negro, rural and urban, native born and foreign born, and by preparing distinct tables for the following states: Indiana, Massachusetts, Michigan, New Jersey, and New York.

There has been urgent need of an application of mathematico-statistical methods to derive reliable results on mortality among the general population, and among various classes of that population. Both from an examination of the present work and through correspondence with Professor Glover, the reviewer is much impressed with the thoroughgoing methods employed in deriving these tables. The results are based on the part of the population of the United States on which we have, in general, more reliable mortality data than on other parts of the country. It is a very important feature of this work that separate tables of infant mortality are prepared, which show the mortality by age intervals of one month for the first year of life, and that the method used in preparing the tables for the first five years of life is the very commendable one used in constructing the German life tables for the decennium 1891–1900. An explanation of the meanings of various columns of the tables is given, with concrete examples, which will make the tables easily understood even by persons who are not particularly familiar with the language in which mortality rates, populations living at given ages, and expectations of life are expressed. One may easily draw from these tables rather striking information in regard to comparative mortality rates of different classes of persons. For

example, of the classes considered, the negro males have the highest mortality, and the white females from the rural population have the lowest mortality. The tables give as the complete expectation of life at birth for the male negro 34.05 years and for the rural female 57.35 years. The average death rate per thousand of the total population of the former is 29.37 while that of the latter is 17.44. As another striking illustration, we note that white males in cities have a complete expectation at birth of 47.32 years and an average death rate per 1,000 of 21.13, while the rural white males have a corresponding expectation of 55.06 years, and an average death rate per thousand of 18.16.

These tables may well become standard for certain civil purposes, such as the valuation of life estates, where there does not exist the selection of lives such as is involved in the data on which life insurance tables are based.

H. L. RIETZ.

NOTES.

THE March number (series 2, volume 18, number 3) of the *Annals of Mathematics* contains the following papers: "Symmetric functions formed by systems of elements of a finite algebra and their connection with Fermat's quotient and Bernoulli's numbers," by H. S. VANDIVER; "The generalized Lagrange indeterminate congruence for a composite ideal modulus," by H. S. VANDIVER; "On the congruence $cx^\lambda + 1 \equiv dy^\lambda$ in a Galois field," by H. H. MITCHELL; "On the geodesics and geodesic circles on a developable surface," by W. C. GRAUSTEIN; "Note on representations of the partial sum of a Fourier's series," by DUNHAM JACKSON; "Acknowledgment," by FRANK IRWIN; "Certain general properties of functions," by HENRY BLUMBERG.

Under an arrangement by which the Mathematical Association of America will contribute to its financial support, the *Annals* will enlarge its annual volume by 100 pages, which will be devoted to expository and historical papers. The subscription price, beginning with volume 19, will be $3.00, with a reduction of one half to members of the Association.

THE November, 1916, number (volume 2, number 11) of the *Proceedings of the National Academy of Sciences* contains

the following mathematical papers: "A set of independent postulates for cyclic order," by E. V. HUNTINGTON; "Certain general properties of functions," by HENRY BLUMBERG. The February and March, 1917, numbers (volume 3, numbers 2 and 3) contain: "Natural and isogonal families of curves on a surface," by JOSEPH LIPKA; "Some problems of Diophantine approximation: the series $\Sigma e(\lambda_n)$ and the distribution of the points $(\lambda_n\alpha)$," by G. H. HARDY and G. E. LITTLEWOOD; "A note on the fitting of parabolas," by J. R. MINER; "The complete enumeration of triad systems in 15 elements," by F. N. COLE, L. D. CUMMINGS, and H. S. WHITE.

AT the meeting of the London mathematical society held on January 18 the following papers were read: By G. H. HARDY and S. RAMANUJAN: "Asymptotic formulas in combinatory analysis"; by M. J. M. HILL: "The singular solutions of ordinary differential equations of the first order"; by H. BATEMAN: "The nature of a moving electric charge and its lines of electric force"; by L. J. ROGERS: "The expansion of the variables of a hypergeometric equation in terms of the ratio of two solutions"; by H. J. PRIESTLEY: "A problem in the theory of diffraction."

THE Swiss mathematical society held its regular summer meeting at Schuls (Engadine) August 6–9, 1916. The following papers were presented: By L. CRELIER: "The power of a line"; by O. SPIESS: "Closure problems for convex curves"; by C. CAILLER: "On imaginary ruled geometry"; by W. H. YOUNG: "Multiple integrals and Fourier series"; by GRACE C. YOUNG: "On the curves of Cellérier and Weierstrass"; by F. RUDIO: "Report on the new edition of Euler"; by M. GROSSMANN: "The completion of the general theory of relativity"; by H. WEYL: "The problem of analysis situs"; by L. G. DU PASQUIER: "On generalized arithmetic"; by G. PÓLYA: "A counterpart of the Liouville approximation theorem in the theory of differential equations"; by O. BERLINER: "On a projective natural geometry"; by K. MERZ: "Historical note on the Steiner surface"; by W. H. YOUNG and GRACE C. YOUNG: "The structure of functions of several variables"; by O. BLOCH: "On the geometry of the Gauss number plane."

AT the annual meeting of the Paris academy of sciences held on December 21, the following prizes were awarded in pure and applied mathematics:

As no satisfactory memoir was received on the problem set in 1915, the Grand prize was not awarded. But a prize of 2,000 francs was given to Professor N. E. NÖRLUND, of the University of Lund, for his contributions to the theory of linear equations in finite differences. The de Parville prize (1500 francs) was awarded to Professor L. TORRES, of Madrid, for his calculating machine for solving algebraic equations and other mechanisms invented by him. The prize in history and philosophy of the sciences (1,000 francs) was given to Professor J. BENSAUDE, of Lisbon, for his book on nautical astronomy in Portugal during the great exploration period. The Francoeur prize (1,000 francs) was awarded to the late Professor L. COUTURAT, of the Collège de France, for his mathematical works. The Poncelet prize (2,000 francs) was awarded to Professor C. J. DE LA VALLÉE-POUSSIN, of the University of Louvain, for his mathematical works. The Lalande, Valz, and Janssen prizes in astronomy were divided among J. E. COGGIA, G. BOCCARDI, CH. FABRY, H. BUISSON, and H. BOURGET.

The following prizes are offered under the usual conditions, the awards to be made in December, 1917: Poncelet prize (2,000 francs) for a memoir in pure mathematics; Francoeur prize (1,000 francs) for the best essay in pure or applied mathematics; Montyon prize (700 francs) for the best practical application of mathematics; Fourneyron prize (1,000 francs) for the most meritorious investigation of ball bearings. The problem set for 1916 is also repeated: the most important improvement of motors used in aviation. The Boileau prize (1,300 francs) for theoretical and experimental progress in hydraulics; de Parville prize (1,500 francs) for a contribution to mechanics; Lalande prize (540 francs) and Valz prize (460 francs) for researches in astronomy; Grand prize (3,000 francs) for the best essay on the problem: "To improve in an important point the study of the successive powers of the same substitution, as the exponent of the power increases indefinitely;" Le Conte prize (50,000 francs) for work in mathematics (one eighth may be used for encouragement, and seven eighths in one or more prizes for results). The general prizes include, for recognition of mathematical accomplishment without more detailed specification, the following: Houllevigue prize (5,000 francs), Wilde prize (4,000 francs) and Jérome Ponti prize (3,500 francs).

THE biennial Ackermann-Teubner prize (1,000 M) has been awarded to Professor E. ZERMELO, of the University of Zürich, for his investigations of the theory of sets, particularly for his memoir of 1907 on well ordered sets.

THE following university courses in mathematics are announced:

COLLÈGE DE FRANCE: (December 1, 1916–March 25, 1917).—By Professor G. HUMBERT: Abelian functions of two variables, two hours.—By Professor J. HADAMARD: Partial differential equations and the problem of Cauchy, two hours.—By Dr. M. BRILLOUIN (from January 2): Internal structure of the earth according to geodesy and seismology, two hours.—By Dr. L. LANGEVIN: Principle of relativity and the theory of gravitation, two hours.

COLUMBIA UNIVERSITY (summer session, July 9–August 17).—By Professor JAMES MACLAY: Fundamental mathematical concepts, five hours; Theory of geometric constructions, five hours; Theory of numbers, five hours.—By Professor EDWARD KASNER: Differential geometry, five hours; Theory of functions of a complex variable, five hours.—By Professor W. B. FITE: Projective geometry, five hours; Higher algebra (Galois theory of equations), five hours.

CORNELL UNIVERSITY (summer session, July 9–August 17).—By Professor V. SNYDER: Foundations of elementary mathematics, five hours; Ruler and compasses, five hours.—By Professor C. F. CRAIG: Higher analysis, five hours.—By Professor F. W. OWENS: Projective geometry, five hours. Professors SHARPE, CARVER, GILLESPIE, HURWITZ and Drs. MCKELVEY and SILVERMAN will conduct a mathematical conference weekly.

UNIVERSITY OF PENNSYLVANIA (summer session, July 9–August 18).—By Professsor G. H. HALLETT: Higher calculus, five hours.—By Professor R. L. MOORE: Foundations of geometry, five hours.—By Dr. F. W. BEAL: Continuous groups, five hours.

UNIVERSITY OF PENNSYLVANIA (academic year 1917–18).—By Professor E. S. CRAWLEY: Higher plane curves, two

hours.—By Professor G. E. FISHER: Functions of a complex variable, two hours.—By Professor I. J. SCHWATT: Infinite series and products, two hours.—By Professor G. H. HALLETT: Finite groups, two hours.—By Professor F. H. SAFFORD: Partial differential equations, two hours.—By Professor M. J. BABB: Theory of numbers, two hours.—By Professor G. G. CHAMBERS: Synthetic projective geometry, two hours.—By Professor O. E. GLENN: Calculus of variations (second semester), two hours.—By Professor H. H. MITCHELL: Algebraic numbers, two hours.—By Professor R. L. MOORE: Foundations of mathematics, two hours.—By Dr. F. W. BEAL: Differential geometry, two hours.

PROFESSOR C. J. DE LA VALLÉE-POUSSIN, of the University of Louvain, has been elected a foreign associate of the Royal society of Naples.

DR. A. TERRACINI has been appointed instructor in analytic geometry in the University of Turin.

PROFESSOR A. M. HARDING, of the University of Arkansas, has been promoted to a full professorship of mathematics.

AT Harvard University the two Benjamin Peirce instructorships have been filled for the coming year by the reappointment of Dr. W. L. HART and the appointment of Dr. T. A. PIERCE. Mr. B. H. BROWN and Mr. J. L. WALSH have been appointed instructors in mathematics. Dr. L. R. FORD, of the University of Edinburgh, has been appointed instructor in actuarial mathematics.

DR. G. R. CLEMENTS, of the University of Wisconsin, has been appointed instructor in mathematics in the U. S. Naval Academy.

PROFESSOR WILLIAM BEEBE, of Yale University, died March 11 at the age of sixty-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

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HANCOCK (H.). Elliptic integrals (Mathematical Monographs, No. 18.) New York, Wiley, 1917. 8vo. 104 pp. Cloth. $1.25

HEDRICK (E. R.). See GOURSAT (E.).

HUNTINGTON (E. V.). The continuum and other types of serial order. 2d edition. Cambridge, Mass., Harvard University Press, 1917. 8vo. 8+82 pp. Cloth. $1.00

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ATWOOD (G. E.). See MYERS (G. W.).

BOREL (E.) et MONTEL (P.). Algèbre. Classe de 3 A, 2 et 1 A B, 3 B, 2 C.D, et enseignement secondaire des jeunes filles. Paris, Colin, 1916. Cartonné. Fr. 3.00

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JOHNS (E. F.). See ANDREWS (P. L.).

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III. APPLIED MATHEMATICS.

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CLEMENTS (G. R.). Problems in the mathematical theory of investment. Boston, Ginn, 1917. 12mo. 24pp. Cloth. $0.32

DAVENPORT (C. B.). Statistical methods with special reference to biological variation. 3d edition revised. New York, Wiley, 1914. 8+225 pp. $1.50

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LIBRARY.

EQUILONG INVARIANTS AND CONVERGENCE PROOFS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society April 24, 1915.)

THE writer has studied the invariants of a pair of analytic curves under the equilong group with the main object of throwing light on the corresponding question in the more important conformal geometry.* The two theories present many analogies, but are not connected by a strict principle of duality. The number of invariants and their orders turn out to be the the same, though the results have to be calculated independently.

In some questions, however, the two theories differ essentially, not only in the methods to be employed, but also in the results obtained. This is true, in particular, with regard to the convergence of the power series entering into the formal calculations. This question was left unsettled in the paper cited.

The principal object of the present paper is to complete the equilong theory by showing that the series in question are always convergent. It thus follows that the equality of the absolute invariants is a sufficient as well as a necessary condition for the equivalence of two pairs of curves. The method used is to reduce the question to one in differential equations† and then to apply certain existence theorems, for solutions at a singular point, due to Briot and Bouquet.

1. *Calculation of the Invariants.*

The equilong group of the plane consists of all contact transformations which convert straight lines into straight lines in such a way that the distance δ between the points of contact of any two curves on a common tangent remains in-

* See Conformal Geometry, Proceedings of the Fifth International Congress, Cambridge (1912), vol. 2, pp. 81–87.

† Such a reduction is impossible in the conformal theory. We have instead a functional equation, and in some cases, as Dr. Pfeiffer has recently shown, the formal solution is actually divergent. In the language of the paper cited above, invariant relations of infinite order are required in conformal equivalence, but not in equilong equivalence.

variant. If we use Hessian line coordinates u, v (where v is the perpendicular distance from the origin and u is the angle which the perpendicular makes with the initial line), the group may be written, in the notation of dual numbers,

$$(1) \qquad U + jV = \text{Function } (u + jv), \quad \text{where} \quad j^2 = 0,$$

or, in separated form,

$$(2) \qquad U = \varphi(u), \quad V = v\varphi'(u) + \psi(u),$$

where φ and ψ are arbitrary analytic functions.* If we change the sign of v, we obtain the improper equilong transformations which preserve the magnitude of δ but reverse its sense.

A single regular analytic curve has no invariants: it can always be reduced to the normal form $v = 0$, that is, the origin (considered as an envelope of lines).

Let us now consider two curves having a common tangent; this tangent we may assume to be the line $u = 0$, $v = 0$. One of the curves we may assume reduced to $v = 0$. The other is defined say by $v = f(u)$, where f is any power series without a constant term. In the second plane let the two curves be written in the form $V = 0$, and $V = F(U)$.

We then have to consider the subgroup of (2) which converts the point $v = 0$ into itself, and the line $u = 0, v = 0$ into itself. This is

$$(3) \qquad U = \varphi(u), \quad V = v\varphi'(u),$$

where $\varphi(u)$ is any power series beginning with the first power of u.

In terms of power series our problem is now as follows: When will the curves

$$(4) \qquad v = \alpha_1 u + \alpha_2 u^2 + \cdots,$$

$$(5) \qquad V = A_1 U + A_2 U^2 + \cdots$$

be equivalent under a transformation of the form

$$(6) \qquad \begin{aligned} U &= a_1 u + a_2 u^2 + a_3 u^3 + \cdots, \qquad (a_1 \neq 0), \\ V &= v(a_1 + 2a_2 u + 3a_3 u^2 + \cdots)? \end{aligned}$$

The requisite condition is expressed by the identity

* See Scheffers, *Math. Annalen*, 1905. The lines and curves considered are oriented.

$$(7)\quad (\alpha_1 u + \alpha_2 u^2 + \cdots)(a_1 + 2a_2 u + 3a_3 u^2 + \cdots)$$
$$= A_1(a_1 u + a_2 u^2 + \cdots) + A_2(a_1 u + a_2 u^2 + \cdots)^2 \cdots.$$

Equating coefficients, we find first $a_1\alpha_1 = a_1 A_1$. Since $a_1 \neq 0$, it follows that $\alpha_1 = A_1$. Hence α_1 is an absolute invariant. In fact α_1, in the curve (4), is the distance from the origin along the tangent $u = 0$, $v = 0$ to the point of contact with the curve. This verifies the invariance of δ, the tangential distance of two curves.

Equating coefficients of u^n in (7), we find an equation involving $a_1, a_2, \cdots, a_n$, the coefficient of a_n being

$$n\alpha_1 - A_1 = (n-1)\alpha_1 \qquad (n > 1).$$

Hence, if we assume $\alpha_1 \neq 0$, the equation can be solved for a_n. Hence no higher absolute invariants exist.

THEOREM I. *A pair of curves whose tangential distance δ is not equal to zero has no absolute invariant, under the equilong group, except δ.*

To show that two pairs of curves having the same δ are actually equivalent, it is of course necessary to show that the first of the series (6), whose coefficients a_n are calculated as described above, is convergent. This we shall do later.

We consider next the case $\delta = 0$, that is, the case where the two curves form a horn angle.* Let the order of contact of the two curves be $h - 1$, where h may be 2, 3, $\cdots$. In our reduced form one of the curves is the point $v = 0$, so the other must be of the form

$$v = \alpha_h u^h + \alpha_{h+1} u^{h+1} + \cdots \qquad (\alpha_h \neq 0).$$

This takes the place of (4). In the second plane the curve (5) must take the same form, since order of contact is obviously an arithmetic invariant. The transformation (6) remains the same.

The first equation obtained from the identity (7) is now

$$\alpha_h = a_1^{h-1} A_h,$$

which determines a_1.† In the next equation the coefficient of

* See the author's paper cited above.

† If $h-1$ is even, and if α_h and A_h have opposite signs, the a_1 thus found will be imaginary. In this case we may apply a preliminary improper equilong transformation, say $U' = U$, $V' = -V$, which will change the sign of A_h. Hence we can always take a_1 to be real. The other coefficients $a_2, a_3, \cdots$ are found rationally, hence the transformation will be real.

a_2 is $(2-h)\alpha_h$; hence we can solve for a_2 unless $h=2$. In general, the nth equation determines a_n unless $h=n$. Hence for a given order of contact, that is, a given value of h (greater than unity), there is one and only one equation of the set which cannot be solved for one of the coefficients (the one with highest subscript) of the transformation. This particular equation, the hth in the list, together with the previous equations, will enable us to eliminate $a_1, a_2, \cdots, a_{h-1}$, thus giving a relation between the coefficients $\alpha_h, \cdots, \alpha_{2h-1}$ and $A_h, \cdots, A_{2h-1}$ of the two curves. This relation can be separated in the form

$$J_{2h-1}(\alpha_h, \alpha_{h+1}, \cdots, \alpha_{2h-1}) = J_{2h-1}(A_h, A_{h+1}, \cdots, A_{2h-1}),$$

where J is a certain rational function of its arguments. Hence we have an absolute invariant of order $2h-1$.

THEOREM II. *Any horn angle, that is, a pair of curves touching each other, has one and only one equilong invariant. If the order of contact is $h-1$, the order of the absolute invariant J_{2h-1} is $2h-1$.*

If we allow h to take the value unity, the curves will *not* be in contact (we may call this contact of order zero), and the invariant J_1 is merely δ, the tangential distance. It is thus easy to restate Theorem II so as to include Theorem I.

In the case of simple contact, $h=2$, the invariant J_3 is of third order and has the following geometric meaning (not restricted to the canonical form in which one of the curves is reduced to a point)

$$J_3 = \frac{r_1\dfrac{dr_1}{ds_1} - r_2\dfrac{dr_2}{ds_2}}{(r_1-r_2)^2} = \frac{\dfrac{dr_1}{d\theta_1} - \dfrac{dr_2}{d\theta_2}}{(r_1-r_2)^2}.$$

Here r_1 and r_2 denote the radii of curvature of the two curves of the horn angle; ds_1 and ds_2 denote the elements of arc; $d\theta_1$ and $d\theta_2$ denote the changes in the inclination of the tangent. The radii and their rates of change are of course taken at the vertex of the angle.*

* The analogous conformal invariant of a horn angle is (see first citation)

$$I_3 = \frac{\dfrac{d\gamma_1}{ds_1} - \dfrac{d\gamma_2}{ds_2}}{(\gamma_1-\gamma_2)^2},$$

where γ denotes curvature.

2. *Convergence Proofs.*

If two pairs of curves have the same absolute invariant, then it is possible at least formally to find a power series for an equilong transformation converting the one pair into the other. To show that the series thus obtained is always convergent, we restate our equivalence problem (again in its canonical form) in terms of differential equations.

If $v = f(u)$ and $V = F(U)$ are to be equivalent under the transformation $U = \varphi(u)$, $V = v\varphi'(u)$, then

$$\varphi'(u) = \frac{F(\varphi(u))}{f(u)}.$$

Here φ is the unknown function. It will be convenient to replace u by x, and φ by y. Thus our differential equation is

$$\frac{dy}{dx} = \frac{F(y)}{f(x)} = \frac{A_1y + A_2y^2 + \cdots}{\alpha_1x + \alpha_2x^2 + \cdots}. \tag{8}$$

The curves will be equivalent if, and only if, this differential equation admits an analytic solution of the form

$$y = a_1x + a_2x^2 + \cdots \qquad (a_1 \neq 0). \tag{9}$$

The formal conditions are obtained from an identity which is obviously the same as (7) with u replaced by x. Hence we have a single condition on the coefficients in (8), namely, $J(\alpha) = J(A)$. The series (9) then formally exists, in fact there will always be ∞^1 such series, one of the coefficients (namely a_h) being arbitrary.

To show that the series obtained are convergent (that is, that the radius of convergence is greater than zero), we might use directly Cauchy's method of majorants; but this is unnecessary, since we can appeal to the following result due to Briot and Bouquet:* If an equation of the form

* Goursat, Cours d'Analyse, vol. 2 (second edition), pp. 503, 504. It is there shown that if the coefficient a_{01}, or b in Goursat's notation, is not a positive integer, there is one solution; if b is a positive integer, there will be either 0 or ∞^1 solutions. But divergent series never arise.

This is of course not true for differential equations of all forms. For example,

$$\frac{dy}{dx} = \frac{y - x}{x^2}$$

is satisfied formally by $y = x + x^2 + 2!x^3 + 3!x^4 + \cdots$, which is divergent for all values of x other than zero.

$$(10)\qquad \frac{dy}{dx} = \frac{a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots}{x},$$

can be solved formally by a power series

$$y = c_1x + c_2x^2 + \cdots,$$

this series will necessarily be convergent.

We discuss first the case where the tangential distance δ is not zero. Then $\alpha_1 = A_1 \neq 0$, so we may write (8) in the form

$$(11)\qquad \frac{dy}{dx} = \frac{(A_1y + \cdots)(\alpha_1 + \alpha_2x + \cdots)^{-1}}{x} = \frac{y + R(x, y)}{x},$$

where R is a power series beginning with terms of the second degree in x and y. This equation is of the form (10). Hence convergence is assured.

We take next the case $\delta = 0$. If the order of contact is $h - 1$, the differential equation (8) becomes

$$(12)\qquad \frac{dy}{dx} = \frac{A_hy^h + A_{h+1}y^{h+1} + \cdots}{\alpha_hx^h + \alpha_{h+1}x^{h+1} + \cdots}, \quad (\alpha_h \neq 0, A_h \neq 0, h > 1).$$

To reduce this to the Briot and Bouquet form, we make a change in the dependent variable, using the substitution

$$(13)\qquad y = (z + \lambda)x, \quad \text{where} \quad \lambda^{h-1} = \frac{\alpha_h}{A_h}.$$

The transformed equation is found to be

$$(14)\qquad \frac{dz}{dx} = \frac{(h - 1)z + R(x, z)}{x},$$

where the power series R starts with terms of the second degree. This is of the form (10), the coefficient b or a_{01} being the integer $h - 1$. Hence z will be a convergent power series. By (13), the same will then be true of y.

THEOREM III. *The convergence of the series defining the equilong transformation is thus guaranteed in every case.*

In our discussion we have assumed one of the curves of the pair in each plane to be reduced to a point, the origin. We may pass directly from this canonical form to the general form, and state our final result as follows:

THEOREM IV. *If one pair of curves*

$$v = \alpha_1 u + \alpha_2 u^2 + \cdots,$$

$$v = \beta_1 u + \beta_2 u^2 + \cdots,$$

is to be equivalent, under the equilong group, to a second pair of curves

$$V = A_1 U + A_2 U^2 + \cdots,$$

$$V = B_1 U + B_2 U^2 + \cdots,$$

the necessary and sufficient condition is the equality of a single absolute invariant J, that is,

$$J(\alpha, \beta) = J(A, B).$$

If the order of contact of the curves of each pair (this is obviously an arithmetic invariant) is $h - 1$, the invariant J is of order $2h - 1$.

If $h = 1$ (curves not touching), J is the tangential distance δ. If $h = 2$ (simple contact), J is a combination of the radii of curvature and their rates of variation, as given above.

COLUMBIA UNIVERSITY,
NEW YORK.

THE INVERSION OF AN ANALYTIC FUNCTION.

BY DR. SAMUEL BEATTY.

(Read before the American Mathematical Society, April 28, 1917.)

THE demonstration of the existence of the inverse of an analytic function is made to depend in the Weierstrass theory upon the power series representation of the function and in the Cauchy theory upon the Jacobian of the real and imaginary parts of the function with reference to the real and imaginary parts of the variable. The proof presented in the following pages finds its source in the Goursat conception of an analytic function and is related as to method to the theory of sets of points.

Suppose the function $w = f(z)$ exists and has a finite derivative at each point of a simply connected domain D.

Suppose z_0 is a point of D for which the derivative is different from zero. There is a simple, closed curve C lying entirely in D and containing z_0 as an interior point, such that for points within or on it the values of the function are all different from one another and the values of the derivative are all different from zero. For if $f(z) - f(z') = 0$ can be satisfied by different values z, z' both as near z_0 as desired, then

$$\begin{aligned} f'(z_0) &= \lim_{z,\, z' \to z_0} \int_\Gamma \frac{f(t)dt}{(t-z)(t-z')} \\ &= \lim_{z,\, z' \to z_0} \frac{f(z) - f(z')}{z - z'} \\ &= 0, \end{aligned}$$

the simple closed curve Γ lying entirely in D and containing within it z_0 and the points z, z' as above described; while if the derivative equals zero for values z as near z_0 as desired, then its value at z_0 is also zero. Denote the aggregate of interior points of C by $\{z\}$ and the aggregate of function values resulting therefrom by $\{w\}$. Denote by $\{z\}'$ the derived set of $\{z\}$ and by $\{\alpha\}$ the aggregate of points of $\{z\}'$ not in $\{z\}$. Then $\{z\}' = \{z\} + \{\alpha\}$. But $\{w\}$ is included in $\{w\}'$, for if $\bar{w}$ is an isolated point of $\{w\}$ and if $\bar{z}$ is the corresponding point of $\{z\}$ then $(w - \bar{w})/(z - \bar{z})$ approaches infinity as z approaches $\bar{z}$, which contradicts the assumption of a finite derivative at $\bar{z}$. Denote by $\{\beta\}$ the aggregate of points of $\{w\}'$ not in $\{w\}$. Then $\{w\}' = \{w\} + \{\beta\}$.

On the basis of the relation $w = f(z)$ there is a (1, 1) correspondence between the points of $\{\beta\}$ and $\{\alpha\}$. For suppose α is a point of $\{\alpha\}$ defined by the sequence $z_1, z_2, z_3, \cdots$ made of points of $\{z\}$ and suppose α is the only limit point of the sequence. The corresponding sequence $w_1, w_2, w_3, \cdots$, made up of points of $\{w\}$ must define at least one number b. If the latter sequence has more than one limit point, then by dropping out corresponding elements from each sequence two new sequences are secured defining only the numbers α and b respectively. Proceeding from such sequences it appears that

$$\lim_{n\to\infty} \frac{w_n - f(\alpha)}{z_n - \alpha} = f'(\alpha) \neq 0, \quad \lim_{n\to\infty} z_n - \alpha = 0,$$

from which it follows that $\lim_{n\to\infty} w_n - f(\alpha) = 0$, which associated with the relation $\lim_{n\to\infty} w_n - b = 0$ furnishes as result $b = f(\alpha)$. It is now evident that there was no possibility of the latter sequence defining more than one number. Hence corresponding to any number α of $\{\alpha\}$ is one and only one number β of $\{\beta\}$ so that $\beta = f(\alpha)$. It may be proved in like manner that corresponding to any number β of $\{\beta\}$ is one and only one number α of $\{\alpha\}$ so that $\beta = f(\alpha)$. Finally this same method of proof coupled with the fact that $\{\alpha\}$ includes $\{\alpha\}'$ may be employed to establish that $\{\beta\}$ includes $\{\beta\}'$.

Each point of $\{w\}$ is an inner point of $\{w\}$. For denote the aggregate of all points in the plane not included in $\{w\}+\{\beta\}$ by $\{\theta\}$. Each point of $\{\theta\}$ is an inner point of $\{\theta\}$, since the limit points of $\{w\} + \{\beta\}$ are all included in $\{w\} + \{\beta\}$. It has been proved that no point of $\{w\}$ is a limit point of $\{\beta\}$ and it remains to prove that no point of $\{w\}$ is a limit point of $\{\theta\}$. Suppose that w is any point of $\{w\}$. With w as center draw a circle of radius σ, within which is no point of $\{\beta\}$. Draw a concentric circle of radius $\sigma/2$. This latter circle contains within it no points of $\{\theta\}$. For if so, suppose $\bar{\theta}$ is such a point. There is then a circle with $\bar{\theta}$ as center and of radius $\bar{\sigma} < \sigma/2$ containing within it only points of $\{\theta\}$ and such that any larger concentric circle contains within it points of $\{w\}$ as well. Such circle must have on its boundary a point $\bar{w}$ of $\{w\}$, whose corresponding point is $\bar{z}$ of $\{z\}$. For z sufficiently near to $\bar{z}$, amp $(z - \bar{z})$ can take values differing by any assigned positive number less than 2π, while amp $(w - \bar{w})$ cannot take values differing by any assigned number greater than π. Hence amp $[(w - \bar{w})/(z - \bar{z})]$ is without a limit as z approaches $\bar{z}$, which contradicts the assumption of a derivative at $\bar{z}$, finite and different from zero. An evident extension would serve to show that there are no points of $\{\theta\}$ within the circle of radius σ.

Moreover, any two points of $\{w\}$ can be joined by a simple, regular curve composed entirely of points of $\{w\}$, for the corresponding points of $\{z\}$ can be joined by a simple, regular curve composed entirely of points of $\{z\}$, which on the basis of the relation $w = f(z)$ furnishes the simple, regular curve desired joining the two points of $\{w\}$. The set of points $\{w\}$ is therefore a domain and the set of points $\{z\}$ is connected therewith by a relation of the form $z = \phi(w)$. The

fact that $\phi'(w)$ exists and equals the reciprocal of $f'(z)$ justifies the conclusion that $z = \phi(w)$ is an analytic function of w for the domain $\{w\}$, and this is the inverse of $w = f(z)$ for the domain $\{z\}$.

Suppose next that the derivative is zero at z_0. It is not possible that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^n} = 0 \quad (n = 1, 2, 3, \cdots),$$

unless $f(z)$ is a constant. This is at once evident from the power series representation for $f(z)$, but it is more in line with the subsequent course of the present paper to establish it in the manner now to be described. Where $n - 1$ is any assigned positive integer and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^{n-1}} = 0,$$

define $w_{n-1}(z)$ in D to equal

$$\frac{f(z) - f(z_0)}{(z - z_0)^{n-1}}$$

for z different from z_0 and to equal zero for $z = z_0$. Then $w_{n-1}(z)$ is analytic in D. If C is any simple, closed curve in D containing z_0 as an interior point, then for any other interior point z

$$\frac{w_{n-1}(z)}{z - z_0} = \frac{1}{2\pi i} \int_C \frac{w_{n-1}(t)dt}{(t - z)(t - z_0)},$$

from which it follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^n} = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t - z_0)^{n+1}}.$$

The proposition is, therefore, to show that it is impossible to have

$$\int_C \frac{f(t)dt}{(t - z_0)^{n+1}} = 0 \quad (n = 1, 2, 3, \cdots),$$

unless $f(z)$ is a constant. If z is any interior point other than the center z_0 of a circle lying entirely within C and if all the integrals above are supposed to be zero, then

$$(z - z_0)f(z_0) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_C \left(\frac{z - z_0}{t - z_0}\right)^n f(t)dt$$
$$= \frac{1}{2\pi i}\int_C \sum_{n=1}^{\infty}\left(\frac{z - z_0}{t - z_0}\right)^n f(t)dt$$
$$= (z - z_0)f(z).$$

That is, $f(z) = f(z_0)$, which contradicts the assumption here made that $f(z)$ is not a constant. Hence for a certain positive integral value of $n - 1$

$$\lim_{z\to z_0}\frac{f(z) - f(z_0)}{(z - z_0)^{n-1}} = 0, \quad \text{but} \quad \lim_{z\to z_0}\frac{f(z) - f(z_0)}{(z - z_0)^n}$$

either does not exist or if so is not zero. The previous argument shows that this latter limit exists and equals

$$\frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t - z_0)^{n+1}},$$

which may be denoted by $\rho e^{i\theta}$. Adopting the definition already given for $w_{n-1}(z)$ it appears that

$$w'_{n-1}(z_0) = \lim_{z\to z_0}\frac{w_{n-1}(z)}{z - z_0} = \rho e^{i\theta}.$$

Substituting ξ^n for $w - f(z_0)$, then for any point z of D other than z_0

$$\left(\frac{\xi}{z - z_0}\right)^n = \frac{w_{n-1}(z)}{z - z_0}.$$

Since

$$\lim_{z\to z_0}\frac{f(z) - f(z_0)}{(z - z_0)^n}$$

exists and is different from zero, z_0 is not a limit point of points z for which $f(z) - f(z_0) = 0$. Since for any point z of D other than z_0

$$f'(z) = (z - z_0)^{n-1}\left[w'_{n-1}(z) + (n - 1)\frac{w_{n-1}(z)}{z - z_0}\right],$$

z_0 is not a limit point of points z for which $f'(z) = 0$, for in that case $w_{n-1}(z_0)$ would equal zero. Hence where d is any fixed positive number less than $\rho \sin(\pi/n)$, a domain R about

but not inclusive of z_0 exists so that for any point z therein ξ^n and dw/dz are both different from zero, and

$$\left| \frac{w_{n-1}(z)}{z - z_0} - \rho e^{i\theta} \right| \leqq d,$$

or what is the same thing,

$$\frac{w_{n-1}(z)}{z - z_0} = \rho e^{i\theta} \left(1 + \frac{h}{\rho} e^{i\phi} \right),$$

in which $0 \leqq h \leqq d$. One of the values of $\xi/(z - z_0)$ corresponding is

$$\rho^{1/n} e^{i(\theta+2q\pi)/n} \left(1 + \frac{h}{\rho} e^{i\phi} \right)^{1/n} = \rho^{1/n} e^{i(\theta+2q\pi)/n} \left(1 + \frac{k}{\rho} e^{i\psi} \right),$$

in which $0 \leqq k \leqq h$. Hence for such a determination of $\xi/(z - z_0)$ it appears that

$$\left| \frac{\xi}{z - z_0} - \rho^{1/n} e^{i(\theta+2q\pi)/n} \right| < \rho^{1/n} \sin \frac{\pi}{n}.$$

Therefore, for z in the domain R there are n determinations of ξ so that the corresponding values of $\xi/(z - z_0)$ lie one within each of n equal circles, each circle lying exterior to the remaining $n - 1$ circles and the centers of the circles representing the nth roots of $\rho e^{i\theta}$. Having regard to the values of $\xi/(z-z_0)$ lying within a definite one, but nevertheless any one of these circles as determined by the values of z in R, and noting that $\xi = 0$ for $z = z_0$, then for the domain composed of R and the point z_0, ξ is a one-valued function of z. Moreover, the derivative of ξ for $z = z_0$ is the number represented by the center of the particular circle chosen. For any point z of R

$$\frac{d\xi}{dz} = \frac{1}{n} \frac{dw}{dz} \xi^{1-n},$$

which is finite and different from zero. That is, ξ is an analytic function of z for the domain R and the point z_0, and for this latter point the derivative is not zero. Hence by decreasing d, a domain R can be found so that the corresponding values of ξ are all different from one another. Hence corresponding to such a domain R and the point $z = z_0$ is a domain S and the point $\xi = 0$, the variables of which are connected by the analytic relations

$$\xi = \phi(z), \quad z = \psi(\xi).$$

It is clear that corresponding to any point w in the vicinity of $f(z_0)$ the function $z = \psi(\xi)$ furnishes n values of z. Also the form of $\psi(\xi)$ would depend on the particular circle chosen, but one form may be transformed into any other by replacing ξ by the product of ξ and the appropriate nth root of unity.

UNIVERSITY OF TORONTO.

EMORY McCLINTOCK.

BUT few members of the American Mathematical Society at the present time appreciate the magnitude of the services rendered by its former president, Emory McClintock, who died July 10, 1916.

He was born September 19, 1840, at Carlisle, Pa. His father was the Rev. John McClintock, a learned Methodist Episcopal clergyman, for a time professor of mathematics, Latin, and Greek in Dickinson College, and during the Civil War chaplain of the American Chapel in Paris. He is perhaps best known as the author, with another, of a "Cyclopædia of Biblical, Theological, and Ecclesiastical Literature."

McClintock went to school for the first time at the age of thirteen, and a year later entered the freshman year of Dickinson College. In 1856, when his father left Dickinson College for New York, he entered Yale, and in 1857 he entered Columbia as a member of the class of 1859. His remarkable ability excited the admiration of his teachers, Professors Charles Davies and William Guy Peck. In April, 1859 in order to meet an emergency caused by the illness of a member of the teaching staff, he was graduated and appointed tutor in mathematics. Soon afterwards his father took charge of the American Chapel in Paris, and in 1860 McClintock resigned his position at Columbia to go abroad. In 1861 he studied chemistry at the University of Gottingen. In 1862 he returned to America with the intention of engaging in the Civil War. He was offered an appointment as second lieutenant of topographical engineers in the United States Army, but on his way to Washington suffered a sunstroke which prevented him

from entering the army. From 1863 until 1866 he was consular agent of the United States in Bradford, England. During 1866 and part of 1867 he was connected with a private banking firm in Paris. In 1867 he was appointed actuary of the Asbury Life Insurance Company of New York, and in 1871 he became the actuary of the Northwestern Life Insurance Company of Milwaukee, a position that he retained for eighteen years.

Early in 1889 he was appointed actuary of the Mutual Life Insurance Company of New York to succeed Professor W. H. C. Bartlett. With this company he remained until his death, serving as actuary until 1911, and continuing thereafter as consulting actuary. From 1905 until 1911 he was also vice-president and trustee. In the autumn of 1906 serious ill health compelled him to relinquish a part of his activities and in 1911 he retired altogether from active work for the company.

McClintock applied for membership in the New York Mathematical Society in November, 1889, was elected vice-president the following month and president at the annual meeting in December, 1890. He retired from the presidency at the annual meeting in December, 1894, on which occasion he delivered a very remarkable and interesting address entitled: "The Past and Future of the Society."* At the same meeting the Society adopted the following resolution:

"*Resolved,* That the American Mathematical Society on the occasion of the retirement from the presidency of Dr. Emory McClintock expresses its appreciation of the great services rendered by him while presiding officer of the Society and its recognition of the fact that to his initiative were due the broadening of organization and extension of membership which have made the Society properly representative of the mathematical interests of America."

The writer, who in 1888 had been offered a position in the actuarial department of the Mutual Life Insurance Company, and who was secretary of the Mathematical Society in its early days, made the acquaintance of Dr. McClintock shortly after his arrival in New York in 1889. He frequently visited McClintock at "Kemble Hill," his home near Morristown, New Jersey, and soon came to admire his learning, scientific ability, breadth of view, and far-sighted wisdom. McClintock at all times displayed the keenest and most active

* Published in the BULLETIN, vol. 1, pp. 85–94.

interest in every plan for enlarging the scope of the Society and improving the character of its work. It was chiefly through his encouragement and support that the BULLETIN was founded in 1891. His influence and financial assistance led the Society in 1896 to undertake the publication of the papers read at the International Mathematical Congress in Chicago in 1893. His wise advice, patience in conference, and generous assistance were most helpful to the Society in connection with the establishment of the *Transactions* in 1900.

On one or two occasions the writer heard him express regret that he had not followed an academic career, which would have permitted him to give a larger share of his time to research and enjoy those mutually inspiring relations that so often exist between a teacher and his students. However McClintock never failed to stimulate and inspire everyone of scientific aptitude or taste with whom he came in contact; and as one who was not by profession a teacher of the subject, he stands practically alone among notable American contributors to pure mathematics.

McClintock's scientific publications date from 1868, when he began his epoch-making contributions to the theory of life insurance. For an account of his work in this field the reader is referred to the appreciation recently published in the *Transactions* of the Actuarial Society of America.*

His contributions to the literature of pure mathematics began when adequate facilities were first afforded for the publication of the mathematical researches of Americans by the establishment of the *American Journal of Mathematics* in 1878. Immediately thereafter he submitted several papers for publication. His first paper, entitled "An essay on the calculus of enlargement," was an effort to present the theory of finite differences and the differential calculus from a unified point of view. The paper may be regarded as a precursor of recent attempts to consider difference equations as differential equations of infinite order.

His other more important papers were a series of researches on solvable quintic equations published in the *American Journal of Mathematics* and a paper on the theory of numbers published in the third volume of the *Transactions*. During the years 1879 to 1899 McClintock published fifteen papers in the *American Journal of Mathematics*, being for a number of

* Vol. 17, part II, (Oct., 1916).

years the most frequent American contributor not on the editorial staff of the Journal.

When one considers that McClintock made no use of the powerful labor saving machinery which has revolutionized modern analysis, the results obtained by him in his researches on quintic equations, as well as some of his other achievements, appear to indicate a truly wonderful power of manipulation and clearness of vision.

He received the honorary degree of Ph.D. from the University of Wisconsin in 1884, LL.D. from Columbia in 1885, and LL.D. from Yale in 1899. He was president of the Actuarial Society of America from 1895 to 1897. He was an honorary fellow of the American Academy of Arts and Sciences, a member of the London Mathematical Society, and a fellow of the Institute of Actuaries.

A list of McClintock's mathematical publications is given below:

1. "An essay on the calculus of enlargement." *American Journal of Mathematics*, volume 2 (1879), pages 101–161.
2. "A new general method of interpolation." *American Journal of Mathematics*, volume 2 (1879), pages 307–314.
3. "On a theorem for expanding functions of functions." *American Journal of Mathematics*, volume 2 (1879), pages 348–353.
4. "Note on a theorem for expanding functions of functions." *American Journal of Mathematics*, volume 3 (1880), page 173.
5. "On certain expansion theorems." *American Journal of Mathematics*, volume 4 (1881), pages 16–24.
6. "On the remainder of Laplace's series." *American Journal of Mathematics*, volume 4 (1881), page 96.
7. "On the resolution of equations of the fifth degree." *American Journal of Mathematics*, volume 6 (1884), pages 301–315.
8. "Analysis of quintic equations." *American Journal of Mathematics*, volume 8 (1885), pages 45–84.
9. "On the algebraic proof of a certain series." Presented to the Society March 6, 1891. *American Journal of Mathematics*, volume 14 (1891), pages 67–71.
10. "On independent definitions of the functions log (x) and e^x." Presented to the Society March 6, 1891. *American Journal of Mathematics*, volume 14 (1891), pages 72–86.
11. "On lists of covariants." *Bulletin of the New York Mathematical Society*, volume 1 (1892), pages 85–91.
12. "On the computation of covariants by transvection." Presented to the Society January 2, 1892. *American Journal of Mathematics*, volume 14 (1892), pages 222–229.
13. "On the non-euclidian geometry." *Bulletin of the New York Mathematical Society*, volume 2 (1892), pages 21–33.
14. "On the early history of the non-euclidian geometry." *Bulletin of the New York Mathematical Society*, volume 2 (1893), pages 144–147.
15. "Theorems in the calculus of enlargement." Presented to the Society August 14, 1894. *American Journal of Mathematics*, volume 17 (1895), pages 69–80.

16. "A method for calculating simultaneously all the roots of an equation." Presented to the Society August 14 and October 27, 1894. *American Journal of Mathematics*, volume 17 (1895), pages 89–110.
17. "The past and future of the Society." Presidential Address, delivered before the Society December 28, 1894. *Bulletin of the American Mathematical Society*, volume 1 (1895), pages 85–94.
18. "On the most perfect forms of magic squares, with methods for their production." Presented to the Society April 25, 1896. *American Journal of Mathematics*, volume 19 (1897), pages 99–120.
19. "On a solution of the biquadratic which combines the methods of Descartes and Euler." Presented to the Society May 29, 1897. *Bulletin of the American Mathematical Society*, volume 3 (1897), pages 389–390.
20. "Further researches in the theory of quintic equations." Presented to the Society August 17, 1897. *American Journal of Mathematics*, volume 20 (1898), pages 157–192.
21 "A simplified solution of the cubic." Presented to the Society December 28, 1900. *Annals of Mathematics*, series 2, volume 2 (1901), pages 151–152.
22. "On the nature and use of the functions employed in the recognition of quadratic residues." Presented to the Society December 27, 1901. *Transactions of the American Mathematical Society*, volume 3 (1902), pages 92–109.
23. "The logarithm as a direct function." Presented to the Society February 28, 1903. *Bulletin of the American Mathematical Society*, volume 9 (1903), pages 467–469.

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ON FOCI OF CONICS.

BY DR. J. H. WEAVER.

THE study of the properties of conic sections, and of certain related points and lines has afforded ample scope for the faculties of all mathematicians from the Greeks on down through the ages. And not the least important among the points connected with the sections are the foci. It is the object of the present paper to give (I) a short historical sketch of the development of the properties of conics connected with the foci; (II) some of the theorems from Pappus which have a bearing on foci and tangents.

I. *Historical Sketch.*

What name should be connected with the discovery of the foci is still a matter of conjecture. Zeuthen* seems to think

* Geschichte der Mathematik im Alterthum und Mittelalter, Kopenhagen, 1896, p. 211.

that the focus for the parabola was known to Euclid. However, we have no mention of such points or of any of their properties until we arrive at the time of Apollonius (about the middle of the second century B.C.). He proves for us in his Conics, Book III, Props. 45–52,* the following properties of foci for central conics.

I. If Ar and $A'r'$, the tangents at the extremities of the axis of a central conic, meet the tangent at P in r and r' respectively, then

(1) rr' subtends a right angle at each focus S and S';

(2) $\angle rr'S = \angle A'r'S'$ and $\angle r'rS' = \angle ArS$.

II. If O is the intersection of rS' and $r'S$, then OP is perpendicular to the tangent at P.

III. The focal radii of P make equal angles with the tangent at that point.

IV. If from either focus as S, SY be drawn perpendicular to the tangent at any point P, the angle AYA' will be a right angle, or the locus of Y is a circle on AA' as diameter.

V. If C is the center of the conic, a line drawn through C parallel to either of the focal radii of P to meet the tangent will be equal in length to CA.

VI. In an ellipse the sum, and in a hyperbola the difference, of the focal distances from any point on the conic is equal to the axis AA'.

Apollonius does not use or mention in any way the focus for the parabola.

The next mention that we have of foci is given us by Pappus (about the end of the third century A.D.). He gives us the first recorded use and proofs of the focus-directrix definition of conics.† In addition to these Pappus has several other lemmas that have a bearing on foci. Their statement and proof will be given in section II.

But although Apollonius and Pappus have recorded for us the most notable of the properties of the foci, neither of them attached any name to the points in question. That honor was left for Johann Kepler (1571–1630). In his work Ad Vitellionem Paralipomena quibus, Astronomæ Pars Optica Traditur, Francofurti, 1604,‡ he gives a short account of the conic sections, a part of which runs as follows:

* See Conics of Apollonius, ed. Heath, pp. 113–118.

† For a discussion of these theorems see my article, Pappus, "Introductory Paper," BULLETIN, vol. 23, No. 3, p. 134.

‡ See Kepler, Opera Omnia, ed. Frisch, Frankofurti, 1859, vol. II, p. 185.

"There are among these curves certain points of especial consideration, which have a certain definition but no name, unless they usurp for name the definition or some property. For if from these points lines are drawn to the points of contact of tangents to the section, these lines make equal angles with the tangents. . . . We, because of the properties of light and the eye, from the viewpoint of mechanics shall call these points foci. We might have called them centers, because they are on the axis of the section, if authors, in the hyperbola and ellipse, were not accustomed to calling another point the center. In the circle there is one focus, the center. In the ellipse there are two foci equally distant from the center, and more removed in the more acute. In the parabola, one focus is within the section and the other may be considered either within or without the section and removed to an infinite distance from the first focus, so that if a line drawn from this '*cœcus*' focus to a point of the section will be parallel to the axis. In the hyperbola, the external focus becomes nearer the internal focus as the hyperbola becomes more obtuse."

In the above rather free translation of Kepler's remarks two things are to be noted as distinct contributions to geometry: (1) The discovery that the parabola may be considered as having two foci; (2) the formulation of the idea that parallel lines are concurrent at a point at infinity.

Following hard upon the footsteps of Kepler, Girard Desargues (1593–1662) extended the notion of the new doctrine of infinity, used the general method of projection and gave methods for determining the foci of a conic both in a plane and when the conic is in a cone. In the plane case he used the circle defined by Apollonius in his conics.* The method for determining the foci in a cone has been summarized by Chasles as follows:

Given any conic O and a cone through it. Let O' be any section of the cone. Through the vertex V pass a plane parallel to that of O' meeting the plane of O in the line ab. Take any point t on ab and let the chord of contacts of the tangents from t to O meet ab in t'. Also let rr' be any segment of ab which subtends a right angle at V. The two sets of points tt' and rr' constitute two involutions having one segment cc' in common. The polars X, X' of c and c' correspond to the axes of O'. . . . The tangents to O from the points r and the lines from r' to their several points of contact determine on

* See I (1) above.

X an involution, whose double points correspond to the foci of O', since every tangent and its normal are harmonic conjugates to the focal distances of the point of contact.*

Another interesting determination for foci was given by Maclaurin (1698–1746).† It is as follows: Let there be a conic section and let its major axis be TT', and let its auxiliary circle be drawn. Let a tangent be drawn to the conic cutting the auxiliary circle in the points A and B. Let AC be a diameter of the auxiliary circle. Let BC be drawn and let it cut TT' in F. Then F is a focus of the conic.

Poncelet (1788–1867), however, greatly extended the theory of the foci when he worked out his theory of ideal chords, and showed that all circles in a plane pass through the same two points φ and φ' on the line at infinity, and indicated the bearing of these points on the foci of conics.‡ He also shows that an analytic calculation of the foci from their definition gives not two foci for each central conic, but four, two real ones on the major or transverse axis and two imaginary ones on the minor or conjugate axis.§ Plücker (1801–68) extended the ideas of Poncelet on foci to plane curves of all orders, regarding as a focus of any curve the point of intersection of any two tangents drawn to it from φ and φ', one from each.‖

A great many properties of foci and methods for determining them have been discovered in modern times. An extensive list of references bearing on this subject may be found in Taylor's Conics Ancient and Modern and in the Encyklopädie der Mathematischen Wissenschaften, Band III_2, Heft I, pages 52–56. Only one of the theorems will be noted here. It is as follows: If any cone is cut by a plane and spheres are inscribed in the cone tangent to the plane, these will touch the plane in the foci of the conic section. One would expect to find such a theorem among those enunciated by the Greeks, but it was not stated in definite form until discovered by Dandelin in 1822.¶

* For a further discussion of the work of Desargues see C. Taylor, "Conics, Ancient and Modern," Cambridge, 1881, pp. lxi and 261, where references to the original sources may be found.

† Geometrica Organica, sive descriptio linearum curvarum universalis, London, 1720, Sec. III, p. 102. See also Poncelet, Traité des Propriétés projectives des Figures, Paris, 1865, Tome I, p. 249.

‡ See Traité, sections 89–98, 453 of tome I.

§ *Annales de Mathématiques*, vol. 8 (1818), p. 222.

‖ *Crelle's Journal*, vol. 10, pp. 84–91.

¶ *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-lettres de Bruxelles*, tome II, pp. 171–202.

II. *Some Theorems of Pappus which have a Bearing on Foci.*

In addition to the above mentioned theorems of Pappus relative to the subject the following ones have some interesting properties which have a bearing on the foci. They are lemmas on the books of Determinate Section of Apollonius.*

THEOREM 1: "Let there be a semicircle AEB on the diameter AB (Fig. 1), and in this the perpendiculars CE and DZ and

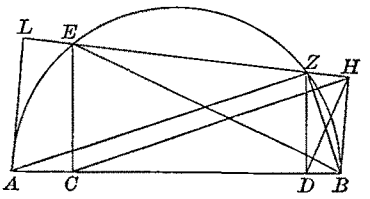

FIG. 1.

let the line EZH be drawn and to this the perpendicular BH, then three things follow: $CB \cdot BD = BH^2$, $AC \cdot DB = ZH^2$ and $AD \cdot BC = EH^2$.

"Let HC, HD, AZ and ZB be drawn. Since $\angle AZB$ is right and DZ is perpendicular to AB, then $\angle DZB = \angle BAZ$. But since the angles BDZ and ZHB are right, the points D, Z, H and B are concyclic, and $\angle DZB = \angle DHB$; then, EB being drawn, $\angle BAZ = \angle BEZ$, being angles in the same segment BZ; and in the segment BH $\angle BEZ = \angle BCH$, therefore $\angle DHB = \angle BCH$. Therefore the triangles CBH and HBD are similar and

$$CB : BH = BH : BD \quad \text{or} \quad BC \cdot BD = BH^2.$$

But also $AB \cdot BD = BZ^2$, whence if we subtract $BC \cdot BD = BH^2$, there remains $AC \cdot BD = ZH^2$. Again since $AB \cdot BC = BE^2$, if we subtract $CB \cdot BD = BH^2$, there remains $AD \cdot CB = EH^2$."

In this proof the equation $BC \cdot BD = HB^2$ signifies that HB is tangent to a circle which passes through C and D and has its center at the midpoint of EZ. If we extend EZ to L, then from the equation $AC \cdot BD = ZH^2$ we have $LE = ZH$, and the circle through D, C and H passes through L. We may then consider AB as the tangent to an ellipse whose major axis is HL, whose foci are E and Z and whose auxiliary circle is the circle through C, D, L and H, while AB is the

* See Pappus, ed. Hultsch, Book VII, Prop. 59, 61, 62 and 64.

portion of a tangent cut off between the two tangents at the vertices of the ellipse, and the circle on AB as diameter is then the circle mentioned by Apollonius as passing through the foci of an ellipse.

THEOREM 2: "Three straight lines AB, BC and CD are given (Fig. 2); then if $AB \cdot BD : AC \cdot CD = BE^2 : EC^2$,

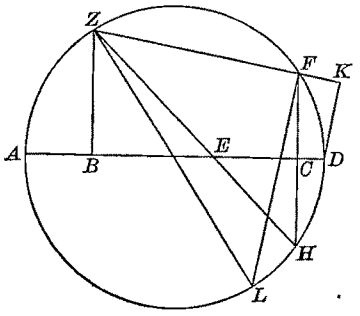

FIG. 2.

$AE \cdot ED : BE \cdot EC$ is a unique and minimum ratio and equals $AD^2 : (\sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)})^2$.

"Let there be described on the diameter AD a circle, and from the diameter let the perpendiculars BZ and CH be drawn to the circumference. Since by hypothesis $AB \cdot BD : AC \cdot CD = BE^2 : EC^2$, and since $AB \cdot BD = BZ^2$ and $AC \cdot CD = CH^2$, then $BZ^2 : CH^2 = BE^2 : EC^2$, and so $BZ : CH = BE : EC$. Therefore the triangles ZBE and HCE are similar and $\angle ZEB = \angle HEC$, and therefore the line through Z, E and H is straight. Let ZEH be drawn and let HC be produced to F, a point of the circumference, and let ZF be drawn and produced to K, whence a perpendicular KD is drawn to ZK; then by the above theorem 1, $AC \cdot BD = ZK^2$, and $AB \cdot CD = FK^2$, therefore ZF (that is $ZK - EK$) $= \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)}$. Now let the diameter ZL be drawn, and FL also; then $\angle ZFL = \angle ECH$, and in the segment ZF, $\angle ZLF = \angle ZHF$. Therefore the triangles ZFL and ECH are similar. Therefore $LZ : ZF = HE : EC$; that is, because LZ and AD are diameters, $AD : ZF = HE : EC$, and so $AD^2 : ZF^2 = HE^2 : EC^2$; that is, on account of the similar triangles HCE and ZBE which make $HE : EC = ZE : EB$, $AD^2 : ZF^2 = HE \cdot ZE : EC \cdot EB = AE \cdot ED : BE \cdot EC$. And

the ratio $AE \cdot ED : BE \cdot EC$ is unique and minimum, and we have shown above that $ZF = \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)}$; therefore $AE \cdot ED : BE \cdot EC = AD^2 : (\sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)})^2$ is a unique and minumum ratio."

In this theorem as in the above, AD may be considered as a tangent to an ellipse which has Z and F for foci and K for a vertex. Moreover E is the point of contact of AD with the ellipse. This is evident if we notice that $\angle ZEB = \angle HEC = \angle CEF$. Pappus, however, does not prove that the ratio is actually a minimum. Fermat does this in his theory of maximum and minimum, where he remarks that Pappus recognized the difficulty of solving such problems.*

THEOREM 3: "Again let there be three given lines AB, BC and CD (Fig. 3); then if $AD \cdot DB : AC \cdot CB = DE^2 : EC^2$,

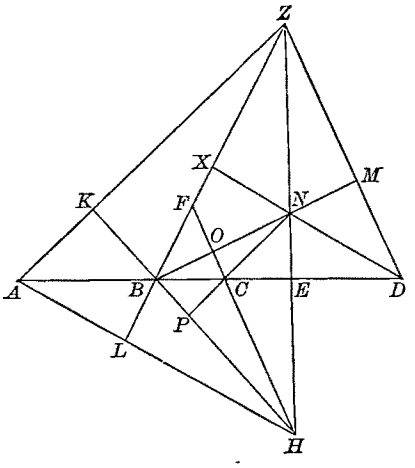

FIG. 3.

$AE \cdot EB : CE \cdot ED$ is a unique and minimum ratio and is equal to $(\sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)})^2 : DC^2$.

"Let the perpendicular EZ be drawn from E to AD and produced so that $AD \cdot DB = ZD^2$ and let HC be drawn parallel to ZD. Since by hypothesis $AD \cdot DB : AC \cdot CB = DE^2 : EC^2$ and in the similar triangles ZED and HEC, $DE : EC = DZ : CH$, then $AD \cdot DB : AC \cdot CB = DZ^2 : CH^2$; and because of the construction $AD \cdot DB = DZ^2$, $AC \cdot CB$

* See Oeuvres de Fermat, Paris, 1896, vol. 3, p. 134.

$= CH^2$. Let AZ, ZB, AH, HB be drawn and let CH be produced and cut the line BZ in F. Since $AD \cdot DB = DZ^2$, $\angle BZD = \angle ZAB$. But because $AC \cdot CB = CH^2$, or $AC : CH = CH : CB$, the angles BHC and BAH are equal. But since DZ and HF are parallel, $\angle BZD = \angle BFH$. Let HB and ZB be produced, and let HB cut AZ in K and ZB cut AH in L; then $\angle BFH + \angle BHF = \angle KBZ$. Moreover $\angle BFH = \angle BZD = \angle ZAB$ and $\angle BHF$ or $BHC = \angle BAH$; therefore $\angle ZAB + \angle BAH = \angle KBZ = \angle KAL$. But $\angle KBL + \angle KAL =$ two right angles, and so the points A, L, B, and K are on a circle. Therefore the angles at L and K are right.* Now let BM be drawn perpendicular to ZD, and let it cut the line EZ in N, and let DN be drawn and produced to X, a point of BZ. Then DX is perpendicular to ZL (see footnote below) and is perpendicular to HL. Let HF cut BN in the point O; then HO is perpendicular to BM (for ZD is perpendicular to BM, and by construction HF is parallel to ZD). Now $AC \cdot CB = CH^2$; therefore $\angle BHC = \angle HAC$. But let NC be produced to P the point of intersection with BH; then NP is perpendicular to BH. Therefore by similar triangles BOH and BPN, the points O, P, H and N are on a circle and in the segment PO $\angle PHO = \angle PNO$, or $\angle BHC = \angle CNB$. Further $\angle HAB = \angle BDN$ in the parallels HA and DX. Therefore if we note that $\angle BHC = \angle CNB$ and $BHC = \angle HAC$, and $\angle HAB = \angle BDN$, whence $\angle CNB = \angle BDN$. Therefore the triangles BNC and BDN with the common angle NBD are similar and in these $DB : BN = BN : BC$ or $DB \cdot BC = BN^2$. But since in the triangle BDZ, DNX is perpendicular, and to this the lines ZN and NB are drawn, then $ZD^2 - DB^2 = ZN^2 - NB^2$. But from construction $AD \cdot DB = ZD^2 = AB \cdot BD + BD^2$; then $ZD^2 - DB^2 = AB \cdot BD$, and so $ZN^2 - NB^2 = AB \cdot BD$. But we have proved that $NB^2 = DB \cdot BC$ and so $ZN^2 = AB \cdot BD + DB \cdot BC = AC \cdot BD$. Therefore $ZN = \sqrt{(AC \cdot BD)}$.

"Again as above $HN^2 - NB^2 = HC^2 - CB^2$. But because as above $AC \cdot CB = HC^2 = AB \cdot BC + BC^2$, $HC^2 - BC^2 = AB \cdot BC$, and so $HN^2 - NB^2 = AB \cdot BC$. But we have shown that $NB^2 = DB \cdot BC$ and so $HN^2 = AB \cdot BC + DB \cdot BC = AD \cdot BC$ or $HN = \sqrt{(AD \cdot BC)}$. Therefore $ZN + NH = \sqrt{(AD \cdot BC)} + \sqrt{(AC \cdot BD)}$.

* Pappus proves this in a preceding lemma. The proof offers no difficulty. See Pappus, Book VII, Prop. 60.

"Again since the angle ZKH is right and AE perpendicular to ZH, by similar triangles AZE and HBE (each being similar to the triangle ABK), then $AE : EZ = HE : EB$, or $AE \cdot EB = ZE \cdot EH$. Therefore $AE \cdot EB : CE \cdot ED = ZE \cdot EH : CE \cdot ED$. But because ZD and HC are parallel, $ZE : ED = HE : EC$ and from this $ZH : CD = ZE : ED$; therefore $ZE \cdot EH : DE \cdot EC = ZE^2 : ED^2 = ZH^2 : CD^2$. Therefore $AE \cdot EB : CE \cdot ED = ZH^2 : CD^2$, and the ratio $AE \cdot EB : CE \cdot ED$ is singular and minimum, and so, since $ZH = \sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)}$, the ratio $(\sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)})^2 : CD^2$ is unique and minimum."

In the above theorem let us assume the line AD to be tangent to a parabola at the point E, and let O be the vertex of the parabola, ON the axis and Z the point in which the normal EN cuts the parabola, and let AZ be the tangent to the parabola at the point Z. Then all the relations in the proof of the above theorem will readily follow.

In theorem 64 Pappus proves that if the points A, B, C, D and E have the order $ABCDE$ on a line then $AD^2 : (\sqrt{(AC \cdot BD)} + \sqrt{(AB \cdot CD)})^2 = AE \cdot ED : BE \cdot EC$ is a unique and maximum ratio and this may be interpreted in terms of the hyperbola in exactly the same way that theorem 2 was interpreted for the ellipse.

Whether Pappus or Apollonius were thinking in terms of conic sections or not in these three problems seems rather doubtful, but if they were it is interesting to note that in the case of the ellipse and the hyperbola use is made of the foci while in the parabola no use is made of any such point. It is also interesting to note that these three are the only maximum and minimum problems (according to Pappus) in the second book of Determinate Section, and that these three cases may be made to depend upon the three conic sections.

West Chester, Pa.

NEW LIGHT ON OUR NUMERALS.

BY MR. JEKUTHIAL GINSBURG.

Introductory Note.—It is interesting to see how much new light is constantly being thrown upon chapters in the history of mathematics which have always been more or less obscure. We know, for example, with reasonable certainty the original habitat of our numerals; we know approximately the century in which they were perfected; we have rather positive information as to the century in which they first appeared in European manuscripts; and we are well advised, through the work of Mr. G. F. Hill, as to their variations in form for the last thousand years. It is true that we do not know when or where the zero of our system was first conceived, although we now have some valuable information as to the one that was used by the Mayas, nor do we know the origin of six of the primitive forms of the digits. Furthermore we do not know with any certainty the date of the first appearance of our numerals on the Mediterranean littoral, but we are not without hope that all this information will sometime be forthcoming, at least to some degree.

Our hope that such further knowledge is not beyond our reach is strengthened by a discovery recently made by M. F. Nau, no report of which seems as yet to have appeared in English. Because of the importance of this discovery, I have asked Mr. Ginsburg to make it known to the readers of the BULLETIN and to supplement the simple statement of the discovery by searching out such information as is available concerning the interesting scholar and teacher, Severus Sebokht, in whose writings the first positive trace of the numerals, outside of India, is found. This he has done, and his article is to my mind particularly valuable because of these features: (1) It shows us that these numerals reached the Arab lands a century earlier than was formerly supposed; (2) it shows that the zero was probably not in the system as then mentioned, showing at least that its value was not generally comprehended in the seventh century and possibly confirming the impression that the symbol had not yet been

invented; (3) it reveals something of the life of a man hitherto unmentioned in the histories of mathematics.

It is to be hoped that this valuable information may prove of such interest to readers that Mr. Ginsburg may be encouraged to tell American scholars, in the near future, something of Sebokht's notable contributions to the study of the astrolabe.

DAVID EUGENE SMITH.

THAT our common numerals are of Hindu origin seems to be a well-established fact,* and that Europe received them from the Arabs seems equally certain, but how and when these numerals reached the Arabs is a question that has never been satisfactorily answered. It is the object of the present article to call the attention of students of the history of mathematics to newly discovered evidence† showing that the Hindu numerals were known to and justly appreciated by the Syrian writer Severus Sebokht who lived in the second half of the seventh century; that is, about a hundred years before the date of the first definite trace that we have hitherto had of the introduction of the system into Bagdad.‡ It will also be shown, on the basis of such information as is available respecting his life and works, that Sebokht was in the most favorable position for getting information of this kind, and that he furthermore had in his possession the most powerful means for the propagating of such knowledge.

Severus Sebokht of Nisibis, bearing the title of bishop, lived in the convent of Kenneshre on the Euphrates§ in the time of the patriarch Athanasius Gammala (who died in 631) and his successor John.‖ He distinguished himself in the studies of philosophy, mathematics, and theology, and in his time the convent of Kenneshre became the chief seat of Greek learning in western Syria. Of his astronomical and geograph-

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* Smith and Karpinski, The Hindu-Arabic Numerals, Boston, 1911.

† By the French orientalist M. F. Nau in the *Journal Asiatique*, series 10, vol. 16 (1910).

‡ Smith and Karpinski, The Hindu-Arabic Numerals, p. 92.

§ W. Wright, Short History of Syriac Literature, London, 1894, pp. 137–139.

‖ Sebokht took part, together with the Jacobite patriarch Theodorus, in a public dispute against the Maronites in the year 659. We have also a letter written by him in the year 665. From these details we may conclude that he flourished in the beginning of the second half of the seventh century. (M. F. Nau, in the *Journal Asiatique*, series 9, vol. 13, p. 60.)

ical works there are a few fragments in a manuscript now in the British Museum.* These fragments consider such questions as whether the heaven surrounds the earth in the form of a wheel or of a sphere; the habitable and uninhabitable portions of the earth; the measurement of the heaven, the earth, and the space between them; and the motion of the sun and the moon. His treatise on the plane astrolabe was published with a French translation by M. F. Nau in the *Journal Asiatique*, series 9, volume 13. Sebokht also wrote a short treatise on eclipses, in which he ridicules the then accepted belief in a celestial dragon as the cause of all such phenomena.†

But the most interesting of Sebokht's writings for the student of history is undoubtedly a fragment of a manuscript‡ published by M. F. Nau, in the *Journal Asiatique* (series 10, volume 16, page 225) in which he directly refers to the Hindu numerals. He seems to have been hurt by the arrogance of certain Greek scholars who looked down on the Syrians, and in defending the latter he claims for them the invention of astronomy. He asserts the fact that the Greeks were merely the pupils of the Chaldeans of Babylon, and he claims that these same Chaldeans were the very Syrians whom his opponents condemn. He closes his argument by saying that science is universal and is accessible to any nation or to any individual who takes the pains to search for it. It is not therefore a monopoly of the Greeks, but is international.

It is in this connection that he mentions the Hindus by way of illustration, using the following words: "I will omit all discussion of the science of the Hindus, a people not the same as the Syrians; their subtle discoveries in this science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have reached the limits of science should know these things they would be convinced that there are also others who know something." This fragment clearly shows that not only did Sebokht know something of the numerals, but

* Add. 14, 538, pp. 153–155.

† See Notes d'Astronome Syrienne, *Journal Asiatique*, series 10, vol. 16 (1910).

‡ Ms., Syriac, Paris No. 346.

that he understood their full significance, and may even have known the zero as Rabbi ben Esra did, in spite of the fact that he, too, speaks of nine numerals. There are two questions that may immediately arise: (1) How could Sebokht have obtained any information about the Hindu numerals? and (2) What are the chances that Sebokht was instrumental in introducing the numerals to the Arabian scholars?

The first of these questions may be answered very easily. Nisibis, the place where Severus lived, was the chief city* of Mygdonia, a small district in the northeast part of Mesopotamia. It was situated in a rich and fruitful country, was long the center of a very extensive trade, and was the great northern emporium for the merchandise of the east and the west. Since the exchange of goods is always accompanied by the exchange of ideas, it is only reasonable to surmise that the different systems of numeration were known in Nisibis, where they could hardly escape the attention of a man like Sebokht, who would surely have been looking for just such information.

The second question is more difficult to answer. It may be said, however, that the weight of the evidence is in favor of Sebokht's work being at least one of the agencies by means of which the knowledge of the numerals was transmitted to the Arabs. He was the head of his convent and occupied a commanding position in the literature of his country. He had many pupils, one of whom, Athanasius of Balad,† was the patriarch of the Jacobites, while such others as Jacob of Edessa‡ and probably George, Bishop of the Arab Tribes,§ were well known as translators and polygraphers. We may be certain that the knowledge of the numerals possessed on the banks of the Euphrates by Severus was transmitted by him to his numerous pupils and through them to other scholars all over Syria. Since we know that Syrian scholars were employed by the caliphs as translators and educators,|| it would be only natural that these Syrians should impart to the Arabs, among other facts relating to the sciences, the knowledge of the Hindu numerals.

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* See Smith's Dictionary of Greek and Roman Geography.
† W. Wright, Short History of Syriac Literature, pp. 154–155.
‡ Ibid., pp. 141–154.
§ Ibid., pp. 156–159; M. F. Nau in the *Journal Asiatique*, series 10, vol. 16.
|| Ernest Renan, Islamisme et la science, p. 9.

SHORTER NOTICES.

Théorie générale des Nombres. By E. DUMONT, Capitaine du Génie Belge. Paris, Gauthier-Villars, 1916. (No. 35 of the series of Scientia.) 93 pp. 2 francs.

IN spite of the distractions on the Yser in 1914–1915, Captain Dumont has been able to produce a very clear account of the fundamental notions of number which he had previously published in extended form in his book Arithmétique générale (1911). This small volume is one of the physico-mathematical series in the Scientia publications of Gauthier-Villars, which undertake to present in succinct form the ideas of science in process of development. Several others of the series have dealt with recent ideas in geometry, analysis, and algebra.

This volume is divided into two parts: Des Nombres absolus, and Des Nombres relatifs. In the first part is given a development of numbers as the ratios of segments of a straight line. The notion of segment of a line is taken as an undefined element. An absolute number is then any law of formation of a segment from a given segment. The product of a segment by a number (absolute number) is another segment. Captain Dumont in fact sets up a set of logical relatives of which the domain is the range of segments of a straight line, the converse domain is the same range, and the relative is considered only with reference to certain processes of addition, subtraction and multiplication of the segments. These processes are supposed to be known. This first part does not differ essentially, save in formal statements, from other derivations of the system of real numbers.

In the second part the fundamental entity is no longer a segment but a vector, by which is meant a directed segment. The notion of direction is assumed. Definitions are set up for free vectors, glissants, symmetric or opposite vectors, and nul vectors. The definition of relative number then follows as any law of determination of one vector from another; by which is really meant again a logical relative whose two domains are the same, namely the range of all vectors. We arrive thus at qualified numbers, which change a glissant vector into another on the same line, or support; versors, which turn glissant vectors around an axis perpendicular to

the line of the vectors; glisseurs, which move a glissant vector parallel to itself along an axis perpendicular to the line of the vector. Combinations of these give quaternions, in which are included complex numbers, vector-quaternions, and surcomplex numbers.

In this part the author insists on the distinction between the vector that is commonly associated with a complex number in the Argand representation, and the number itself. This distinction is, in the opinion of the reviewer also, vital in many places, and unless noticed leads to trouble. For instance, the Steinmetz treatment of alternating currents is handled by Cramp and Smith from this point of view to the improvement of the whole treatment. Hamilton was the first to define a quaternion as the ratio of two vectors, having arrived at this definition by a study of certain groups of linear substitutions, of which one was a form of the quaternion group as an abstract group. The later expositions given by Hamilton used this geometric method of approach. One might go farther and produce a system of numbers corresponding to the ratios of the members of any set of geometric elements, such as bipoints, biplanes, as Cailler and others have done. Some of these geometric elements have even received names, as flèche, bouclier, drapeau, feuillet, moulinet. Captain Dumont, however, develops the notion only as far as biquaternions.

He insists rightly that the method of development is synthetic, and demands no postulates except those of euclidean space, while other methods of the arithmetico-logical type demand large lists of postulates. Further the development is not referred to any coordinate system, and to no fixed base of units. It becomes thus autonomous. Whether he is correct in thinking it could be taught easily from this intuitive point of view is a question which time and experience will settle. The intuitive method seems at least as good as any way to identify the fundamental elements that one starts with, and in a genuinely logical sense it is just as rigorous. When anyone thinks that by talking of the set of elements $A, B, C, \cdots$ he is any better able to identify an element than he is when he talks of the segments of the line, $OA, OB, OC, \cdots$ he is deceiving himself into a fancied rather than a real rigor of treatment. The really essential thing, it would seem, is whether or not some unstated quality of the element is

one that the deductions depend upon. To illustrate, one may prove the theorems of geometry relating to the circle by using wire circles just as well as by using a highly artificial definition of an abstract circle, provided that he does not depend upon the copper or silver, or their qualities, nor upon the thickness, or cross-section of the wire. In most work in mathematics we are engaged in finding the deductions that can be drawn from certain features only of a concrete existence and which are not in the least affected by other features of the object. In other words the most highly concrete object is just as good as the most highly abstract object for deductions that are limited to certain characters possessed in common by the two. From this point of view the definition from geometric objects is justifiable. On the other hand the recognition of the non-geometric character of all the numbers deduced is one that if more common would prevent much wasted ink and time. Such a notion as that of the product of two geometric vectors ought to disappear from the field, for instance, save as a phraseology perhaps. The fruitless discussions over the identification of the right quaternion and the vector (geometric) would no longer be heard of, and all vector systems would be recognized as algebras of hypernumbers.

JAMES BYRNIE SHAW.

Napier Tercentenary Memorial Volume. Edited by CARGILL GILSTON KNOTT. London, Longmans, Green and Company, 1915. xi+441 pp. Price, £1 1s.

THIS sumptuous volume compiled by Dr. Knott is made up of the addresses and essays communicated to the International Congress held in Edinburgh in 1914 to commemorate the tercentenary of the publication of Napier's epoch-making Mirifici Logarithmorum Canonis Descriptio.

The papers are both historical and mathematical, the former dealing with the life and works of Napier and of his contemporaries, immediate predecessors, or followers, and the latter part of the work treating of the modern progress in calculation, in the preparation of tables, and the like. There is also an account of the Edinburgh meeting, with the addresses of a formal congratulatory nature, a list of members, and two indexes, one of subjects and the other of names.

The historical papers are as follows: "The invention of logarithms," by Lord Moulton, a careful study of the working

of Napier's mind in the development of his theory, made with all the acumen of one of the best legal minds of the present time; "Merchiston Castle," by George Smith, then the head-master of Merchiston Castle School, the best description of the building that has been published, illustrated by a colored print; "Logarithms and computation," by Dr. J. W. L. Glaisher, partly historical but chiefly given to the post-napierian development; "The law of exponents in the works of the sixteenth century," by the present reviewer; "Algebra in Napier's day and alleged prior inventions of logarithms," by Professor Florian Cajori, a careful and scholarly investigation of the claims of others with a judicial decision against such claims; "Napier's logarithms and the change to Briggs's logarithms," by Professor G. A. Gibson, a critical study of the important question of the relation between Napier and Briggs in the introduction of the base 10; "The introduction of logarithms into Turkey," by Lieut. Salih Mourad, wherein it is shown that logarithms found their way into Turkey in 1714, and that they were explained in the Turkish language in 1765, through the efforts of Ismail Effendi; "A short account of the treatise 'De arte logistica,'" by Professor J. E. A. Steggall, with a facsimile of one of the pages; "The first Naperian logarithm calculated before Napier," by Professor G. Vacca, in which it is shown that Pacioli (1494), in a problem on compound interest, gave a rough calculation of log 2 in solving what is practically $(1 + r/100) = 2$; but that Pacioli had any real conception of logarithms is not asserted; "The theory of Naperian logarithms explained by Pietro Mengoli (1659)," by Professor G. Vacca, in which it is shown that Mengoli, a pupil of Cavalieri, set forth his explanation in the Geometria Speciosa which was published in Bologna in 1659.

One of the most important articles in the book is the bibliographical chapter by Professor R. A. Sampson on the books exhibited at the time of the meeting. Professor Sampson has given a careful description of each of the early editions, with numerous facsimiles, including two pages from the Aritmetische vnd Geometrische Progress Tabulen of Jobst Buergi, printed in Prague in 1620. These facsimiles show the title page and also one page of the tables. Since this was lent by the Town library of Dantzig, and was to have been returned early in August, 1914, it would be interesting to know where it has been stranded during the last two and a half years.

The mathematical articles include a study of Napier's rules, by Professor Somerville; a consideration of fundamental trigonometric and logarithmic tables, by Professor Andoyer; a description of the work of Edward Sang, by Professor Knott; a study of formulas used in calculation for the development of a function of two variables in spherical harmonics, by Professor Bauschinger; a study of nomograms and of the early calculating machines, by Professor d'Ocagne; a new table of natural sines, by Mrs. Gifford; a consideration of the arrangement of tables, by Dr. J. R. Milne; a note on 'critical' tables, by Mr. T. C. Hudson; a study of economy of entries in tables, by Professor J. E. A. Steggall; the graphical treatment of crystallographic problems, by Dr. A. Hutchinson; a method of computing logarithms by simple addition, by Mr. Schooling; a question as to reducing to a minimum the mean error of tables, by Mr. Erlang; a study of the extension of accuracy of tables by improvement of differences, by Dr. W. F. Sheppard, a paper of unusual interest; a method of finding antilogarithms without tables, by Dr. Artemas Martin; the approximate determination of the functions of an angle, by Mr. H. S. Gay; and a study of life probabilities on the Goldziher criterion, by Mr. Quiquet.

The work is illustrated with great care. The colored portrait of Napier, from the original painting in the University of Edinburgh, is a very welcome addition to the list of portraits of mathematicians now available for study, and the fascimiles from early printed works are of great value.

It is impossible within the limits of a review of this nature to enter into an extended description of the articles. Suffice it to say that there has never appeared a work on the history of any single mathematical discovery that has been so sumptuously published or that contains so much valuable material for the student of the history of mathematics. May we not hope that the forthcoming two-hundredth anniversary of the death of Newton may see a similar meeting in Cambridge, with the result of the publication of a similar volume relating to the life and works of the world's greatest mathematical physicist?

DAVID EUGENE SMITH.

A Course in Mathematical Analysis. By EDOUARD GOURSAT; translated by E. R. HEDRICK and OTTO DUNKEL. Volume II, Part I: *Functions of a Complex Variable.* Boston, Ginn and Company, 1916. v+259 pp.

THOSE who are familiar with Professor Hedrick's excellent translation of the first volume of this standard French treatise will welcome this continuation of his work. To have books of this type available for the use of students who do not read French with facility is a great boon to American teachers. In particular the present volume is admirably adapted for use as a text in an introductory course on the theory of functions of a complex variable.

The content of the book has already been summarized in Professor Osgood's review of the first edition of the original work.* With regard to the merits of the translation, it is sufficient to say that it maintains the high standard already set in the translation of the first volume. The separation into two parts of the translation of volume II increases the availability of the work for use in American universities.

CHARLES N. MOORE.

A Review of High-School Mathematics. By WILLIAM DAVID REEVE and RALEIGH SCHORLING. Chicago, The University of Chicago Press, 1915. x+70 pp.

THIS little handbook has now been out for nearly two years, and has had an opportunity of being put to the test. What the results of such a test are, however, can only be determined by the users. Nevertheless it is evident to anyone that such a handbook must have its chief value in a school where the course of study is the same as that followed in the exercises. The authors have evidently had this in mind, for there are not any very serious departures from the sequence of topics generally adopted in our best American schools. Even with a class which begins algebra in the conventional manner there is no reason why the slight amount of intuitive geometry in the sections devoted to the equation should not be welcome.

On the whole it may be said that the book is modeled somewhat after the exercise manuals so commonly used in Europe. In schools in which the textbooks in use do not supply sufficient review material, which is not generally the case with the leading American textbooks, this manual may be found helpful.

DAVID EUGENE SMITH.

* BULLETIN, vol. 15 (1908–09), pp. 120–126.

Plane Geometry. By JOHN W. YOUNG and ALBERT J. SCHWARTZ. New York, Henry Holt and Company, 1915. 223+x pp.

THE need of a textbook such as Young and Schwartz's Plane Geometry is perhaps best stated by the authors themselves in their preface. "As a result of the widespread discussion during recent years on the improvement of our courses in elementary geometry, the majority of thoughtful teachers appear to have reached substantial agreement on at least one point: To begin the course in plane geometry in the traditional formal manner is pedagogically irrational and scientifically unnecessary. There has accordingly arisen an increasing demand for a textbook which will supply a pedagogically rational approach to the study of plane geometry without sacrificing the logical structure of the subject."

The authors have departed from the orthodox formula for writing plane geometry texts. They do not begin with a formal (and in the eyes of the student also formidable) set of definitions and axioms followed by a proposition of the type: All right angles are equal. Instead the first chapter of 46 pages seeks to make the student familiar with the common geometric concepts. In the second chapter such notions as undefined term, fundamental proposition, geometric proof are clearly explained. Then begins the formal geometry. One finds considerable change in the list of propositions from what one ordinarily sees in an elementary text. Very many theorems which neither serve as links in the logical development nor are important in themselves have been omitted. In the space thus gained the definitions and elementary applications of the trigonometric ratios are introduced.

The authors have made geometry 'easy' in the sense that their explanations and proofs are put in such language as to be intelligible to the average student. The problems are numerous and well selected. Mechanically the book is well gotten up. The use of two color printing for the figures, the auxiliary and construction lines being in a quiet green (not bright red), helps materially. It may be that the reviewer is prejudiced since the book happens to agree with his notions of what an elementary plane geometry text should contain, and it is for that reason that he believes that it is a most welcome and valuable addition to our small list of *good* mathematical textbooks. M. G. GABA.

Ruler and Compasses. By HILDA P. HUDSON, M.A., Sc.D. London, Longmans, Green and Company, 1916. 143 pp. 93 figures. Price $1.80.

THIS little book is a welcome addition to the literature on the boundary between elementary and advanced mathematics. While a number of excellent texts exist in other languages, either originally or by translation, heretofore we have had neither in English. The present claims only to be a compilation from other books and memoirs on the subject, yet it is really much more, as it is full of the ingenious devices peculiar to the British school of mathematics.

With the exception of the well known book of Enriques* it is more systematic than any of the preceding treatises, and excells that one by being more practical and less verbose. In fact, in places the book is too compact to be of maximum service.

The problem is introduced by comparing the postulates and constructions of Euclid and showing that every construction is based on the interpretation of the solution of linear and quadratic equations. This excellent presentation is followed by a survey of the methods by which the various problems are to be attacked; it is unfortunate to bring in a considerable number of new words here, the meaning of which is given later.

The elements of cartesian geometry are developed and applied, but the explanations are too brief. A straight line is assumed tacitly to have a linear equation; the coordinates of the point common to two lines are shown to be expressible rationally in the coefficients in the equations of the lines. The section on the domain of rationality is well written.

The regular polygons are scarcely mentioned. It is stated entirely without proof that the inscription of a regular polygon depends on a binomial equation; the limitation on the exponent of the unknown is clearly derived. The entire construction of the heptadecagon is disposed of in ten lines and a figure.

The treatment of constructions by means of the ruler alone is prefaced by a discussion of projective properties, including vanishing lines, Desargues's theorem, cross ratio, harmonic section, involution, homography, and double elements. Many of these are given entirely without proof, and in no case is the

* Questioni riguardanti la geometria elementare, Bologna, 1902; German translation by Fleischer: Fragen der Elementargeometrie, Leipzig, 1907.

proof full enough to be of much use. On the other hand, the application to elementary metrical cases is much more satisfactory.

Constructions by ruler and compasses bring in the cross ratio of four points on the circle; otherwise it keeps closely in touch with principles developed by Euclid. Then follows a chapter which employs the method of three trials, the idea of displacement, similarity, inversion, and duality, followed by another devoted to special devices on account of limited space, etc.

The last two chapters are given up to constructions made by the ruler and one fixed circle, and by the compasses alone, respectively; each contains a scheme of comparison of the relative merits of different constructions.

To sum up, the immediate practical problem is everywhere well treated, but the foundations, taken from post-euclidean ideas, are by no means so well done. Many of these ideas could have been dispensed with altogether, and the others easily proved for the purpose in hand, under restrictive hypotheses. It is true that this alternative also has objections; if a reader delves further into projective and analytic geometry, he would approach the fundamental theorems under false impressions. To one already familiar with these two subjects, the whole problem becomes an easy one, but to those who are familiar with elementary geometry and algebra only, the choice of the restrictive premises seems to the reviewer to be the wiser procedure.

The style is clear and the figures are well drawn; one unusual feature is that every algebraic equation is written in boldfaced type, making a formidable-looking page. The book is not provided with an index, nor are more than a very few exercises left for the reader to work out.

VIRGIL SNYDER.

Calculus. By HERMAN W. MARCH and HENRY C. WOLFF. (Modern Mathematical Texts, edited by C. S. SLICHTER.) New York, McGraw-Hill Book Company, 1917. 16+360 pp. $5 \times 7\frac{1}{2}$. 125 figs. Price $2.00.

HAVING in mind particularly the needs of technical students, the authors have used for illustrative purposes many practical problems. For the same reason the exercises contain many problems drawn from engineering. The book contains

about the average number of applications. There are other books on the calculus which contain many more applications than the present volume. Two problems are discussed in some detail which are omitted in most calculus books, viz., the cable with a uniform horizontal load (parabolic cable) and the cable with equal loads on equal lengths (catenary).

The authors have exercised considerable care and have met with more than ordinary success in making clear the meaning of an infinitesimal, curvature, and mean value. Duhamel's theorem is explained with unusual clearness and is used consistently in obtaining integral forms. Illustrative examples are numerous.

The preface states that a large number of drill problems have been inserted. In general we find this to be true. None of the lists are excessively long. Some are in our opinion too short. Chapters XVI and XIX end with no problems. We find other chapters ending with 1, 2, 3, and 5 problems. Answers are given to approximately fifty exercises.

The subject of integration is first brought to the student's attention on page 46 just after he has learned to differentiate the power function. Thereafter whenever he is shown how to differentiate a new function he is at once also shown how to apply the result to the integration of certain forms.

In the short sixteen-page chapter on differential equations no use is made of the initial conditions to determine the constants of integration with the exception of the discussion of a damped harmonic motion in the last article of the book. No exercises are given after this discussion. The subjects of center of gravity and moments of inertia have been treated somewhat more fully than is usual.

The book is designed for a course of four hours a week throughout the college year. But it is easy to adapt it to a three-hour course by suitable omissions. On the whole the book is a good one and has added its share to the general endeavor to obtain better texts.

W. V. LOVITT.

Die Differentialgleichungen des Ingenieurs. By Dr. W. HORT, Engineer in the Siemens-Schuckert Works. Berlin, Springer, 1914. 540 pp.

THIS volume cannot fail to be of interest to teachers of mathematics in American schools of engineering. The follow-

ing paragraphs taken from the preface indicate the controlling purpose of the author.

"Da ich möglichst alle für Aufgaben des Ingenieurswesen wichtigen Methoden bringen wollte, habe ich mich veranlasst gesehen, die Reihenentwickelungen nach Frobenius nebst der damit in Zusammenhang stehenden Ermittelung der logarithmenbehafteten Integrale linearer Differentialgleichungen zu behandeln. Bekanntlich werden diese Verfahren in der Behältertheorie gebraucht. Daneben werden einige Fragen, die mit der Technik nicht in unmittelbarem Zusammenhang stehen, wie z. B. die Integration der Differentialgleichungen der Planetenbewegung, ihres allgemeinen Interesses halber erörtert." * * * "Im Interesse der Anwendungen sind auch die Differenzengleichungen wenigstens in einem kurzen Abriss aufgenommen werden."

Then comes the significant statement:

"Beim Abschluss des Druckes ersehe ich aus Nr. 20 der *Zeitung des Vereines deutscher Ingenieure,* dass der deutsche Ausschuss für technisches Schulwesen in seinem fünften Bericht den gleichen Anschauungen Ausdruck gibt, die mir den Abstoss zur Abfassung dieses Buches gegeben haben."

For the reviewer the natural query as to what the book stands for in engineering education in Germany is answered in the paragraph just quoted. One is at first inclined to conclude that the author's point of view is a personal one. In fact, the scope of the book is so comprehensive,—including in the first part (pages 1–253) a thorough course on ordinary differential equations, with applications to a great variety of engineering problems, and in the second part (pages 254–519) an exposition of the differential equations of mathematical physics, with emphasis on such problems as Helmholtz's vortex theory and Lorenz's investigations on turbines,—that the first thought is that the author takes an extremely advanced position as to the possible use a practising engineer may have to make of mathematics. But the authority of the committee on technical education referred to puts the matter in a different light. Evidently the position that engineering students should be trained in mathematics far beyond the calculus receives organized support in Germany.

The book can be used with advantage by American teachers as a storehouse of problems with which to vitalize the subject of differential equations.

PERCEY F. SMITH.

Fundamental Sources of Efficiency. By FLETCHER DURELL. Philadelphia and London, J. B. Lippincott Company, 1914. 368 pp.

IN these latter days of accelerated national development, an era in which whole philosophies of life are being subjected to their supreme test—the right to live—the old principle of the survival of the fittest has assumed a new aspect and we now recognize it in the problem of national efficiency.

We have all read of the wonderful results obtained by men of genius through the intelligent application of a few simple principles of efficiency in concrete fields. The magazines have been flooded with stories using these as themes. Technical journals have presented arrays of facts and figures illustrating the astounding results obtained through their use; efficiency engineering is a modern and live profession; economists are dividing themselves into schools in accordance with their beliefs concerning the workings of these principles and some philosophers tell us that the social order is rapidly changing—that intelligent cooperation is superseding individualism.

We have seen manufacturing plants reorganized; yes, even self-appointed committees of politicians from state legislatures have set themselves the—to them—simple task of reorganizing great universities.

But we have not—at least not all of us—read a book such as that under review, in which are summarized and analyzed into a few elemental principles the various forms and sources of efficiency. It is so written that the general reader may enjoy its pages and a mathematician may easily find in the groups of exercises at the close of the chapters many a mental stimulus. An efficiency expert would find the summaries near the ends of the chapters logical recapitulations of discussions carried out—bird's-eye views of the fields covered therein.

The following are the headings of a few of the 18 chapters and appendices *A* and *B* contained in the book: II. Reuse, III. The unit and its multiplier, IV. The group, · · ·, IX. Expenditure and results, · · ·, XIII. Rhythm. These titles are sufficient to suggest, perhaps outline, the nature of the discussion in each chapter. The principles and sources of efficiency are discussed in detail and a wealth of concrete illustrations cited. Among these are many of the classic applications of which we have read and which have shown in their

simplicity and results that the whole subject is well worth while.

Any instructor might easily learn many a lesson of value from the pages of the book—if he cared to investigate, for example, just how efficiently he is handling a section of students in any subject he may be teaching.

In Chapter XVII the author shows how the various efficiency methods advocated apply to such specimen departments of human thought and endeavor as psychology, education, sociology, business, etc. A brief historical survey is given in Appendix *B*.

ERNEST W. PONZER.

NOTES.

AT the meeting of the London mathematical society on February 1 the following papers were read: By P. A. MACMAHON: "The significance of a certain algebraic function in the theory of distributions" and "The number of ways of pairing off the members of two identical sets of different quantities"; by W. H. SALMON: "Curves of constant torsion."

AT the meeting of the Edinburgh mathematical society on March 9 the following papers were read: By G. B. JEFFERY: "The transformations of axes for Whittaker's solution of Laplace's equation"; by H. DATTA: "On the failure of Heiermann's theorem."

AMONG recent mathematical works published by Gauthier-Villars are the following: Œuvres de Cauchy, 2e série, Tome XII (Nouveaux Exercices d'Analyse et de Physique, Tome 2), Œuvres de Poincaré, Tome II, and two volumes of the Borel series of monographs, Leçons sur les Méthodes de Sturm dans la Théorie des Equations différentielles linéaires et leurs Développements modernes, by Professor MAXIME BÔCHER, and Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire, by Professor C. DE LA VALLÉE POUSSIN. Tome XIII, 2e série (Nouveaux Exercices d'Analyse et de Physique, Tome 3), of the Œuvres de Cauchy is now open to subscription, and Tome I of the Œuvres de Poincaré is in press.

THE Rice Institute, of Houston, Texas, has just issued, in three volumes, the Book of the Opening of the Rice Institute, an account of the academic festival held in celebration of its formal opening in October, 1912. These volumes contain, in addition to an account of the exercises of dedication, the inaugural lectures delivered by the delegates of various institutions. Among these are the following mathematical lectures: "Molecular theories and mathematics," "Aggregates of zero measure," and "Monogenic uniform non-analytic functions—the theories of Cauchy, Weierstrass, and Riemann," by Professor EMILE BOREL, delegate of the University of Paris; and "Henri Poincaré," "The generalization of analytic functions," and "On the theory of waves and Green's method," by Professor VITO VOLTERRA, delegate of the University of Rome.

BY recent action of the Board of Trustees of the University of Chicago, the President of the University, on recommendation of the head of a department, will welcome doctors of philosophy of the University of Chicago and other universities as guests of the university, with the privilege of attending seminars and of carrying on research in the laboratories and libraries. There will be no charge except for laboratory supplies and a nominal laboratory fee where laboratory work is done. Arrangements should be made with the university in advance.

THE Adams prize of Cambridge University has been awarded to J. H. JEANS, former fellow of Trinity College, for his essay: "Some problems of cosmogony and stellar dynamics." The value of the prize is £250.

PROFESSOR A. N. WHITEHEAD has been elected president of the British mathematical association.

MR. P. A. MACMAHON has been elected president of the Royal astronomical society.

PROFESSOR EDWARD KASNER has been elected a member of the National Academy of Sciences.

DR. E. A. T. KIRCHER, of Harvard University, has been appointed instructor in mathematics at the University of Minnesota.

MR. T. R. HOLLCROFT has been appointed instructor in mathematics at Columbia University.

MR. W. H. WILSON, of the University of Illinois, has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

PROFESSOR GASTON DARBOUX, of the University of Paris, and permanent secretary of the Paris Academy of Sciences, died February 22, at the age of 74 years. He was author of a four-volume treatise on differential geometry bearing the title Théorie générale des surfaces, a work on orthogonal systems of surfaces, one on the cyclide and allied surfaces, and a large number of memoirs in the leading mathematical periodicals. During the last thirty years he has been a member of more than fifty learned societies.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAILEY (F. H.). See WOODS (F. S.).

BARROW (I.). Geometrical lectures. Translated with notes and proofs by J. M. Child. London, Open Court, 1917. 8vo. 14+218 pp. 4s. 6d.

BÔCHER (M.). Leçons sur les méthodes de Sturm dans la théorie des équations différentielles linéaires et leurs développements modernes, professées à la Sorbonne en 1913–1914. (Collection de Monographies sur la Théorie de Fonctions publiée sous la Direction de M. Emile Borel.). Recueillies et rédigées par G. Julia. Paris, Gauthier-Villars, 1917. 8vo. 6+118 pp. Fr. 5.00

BOREL (E.). See BÔCHER (M.).

CAJORI (F.). A history of elementary mathematics with hints on methods of teaching. Revised and enlarged edition. New York, Macmillan, 1917. 8vo. 8+324 pp. $1.75

CHILD (J. M.). See BARROW (I.).

JULIA (G.). BÔCHER (M.)

DE MONTESSUS DE BALLORE (R.). Leçons sur les fonctions elliptiques en vue de leurs applications. Paris, Gauthier-Villars, 1917. 8vo. 10+268 pp. Fr. 12.00

REY PASTOR (J.). Fundamentos de la geometría proyectiva superior. (Junta para ampliación de estudios e investigaciones científicas, Laboratorio y seminario matemático publicaciones, Tomo 1.) Madrid, 1916. 8vo. 22+444 pp. Pes. 12.00

SIBIRANI (F.). Riassunto formulario di geometria analytica, algebra, calcolo infinitesimale, calcolo vettoriale e meccanica razionale. Roma, Atheneum, 1915. 8vo. 4+575 pp. L. 12.00

WOODS (F. S.) and BAILEY (F. H.). Analytic geometry and calculus. Boston, Ginn, 1917. 8vo. 12+516 pp. $3.00

II. ELEMENTARY MATHEMATICS.

CAJORI (F.) and ODELL (L. R.). Elementary algebra, second year course. New York, Macmillan, 1916. 9+197 pp. $0.75

DARNELL (A.). See LYMAN (E. A.).

KLAPPER (P.). The teaching of arithmetic. New York, Appleton, 1916. 387 pp. $1.45

LYMAN (E. A.) and DARNELL (A.). Elementary algebra. New York, American Book Company, 1917. 8vo. 8+503 pp.

MILNE (R. M.). Mathematical papers for admission into the Royal military academy and the Royal military college, September–November, 1916. London, Macmillan, 1917. 32 pp. 1s.

ODELL (L. R.). See CAJORI (F.).

III. APPLIED MATHEMATICS.

AKIMOFF (N. W.). Elementary course in Lagrange's equations and their applications to solutions of problems of dynamics with numerous examples. Philadelphia, Philadelphia Book Company, 1917. $2.00

BARNARD (R. J.). Elementary dynamics of the particle and rigid body. London, Macmillan, 1917. 7+374 pp. 6s.

BRAGDON (C.). Projective ornament. Rochester, N. Y., Manas Press, 1915. 79 pp. $1.50

CHOW (H. K.). See HUNSAKER (J. C.).

CLARK (V. E.). See HUNSAKER (J. C.).

DOUGLAS (D. W.). See HUNSAKER (J. C.).

F. G. M. Manuel de mécanique conforme au programme de 1905 et de 1911. Tours, A. Mame, et Paris, J. de Gigord, 1916. 16mo. 16+432 pp.

FURUHJELM (R.). Recherches sur les mouvements propres des étoiles dans la zone photographique de Helsingfors. Helsingfors, Société de Littérature Finnoise, 1916. 190 pp.

GIBSON (G. E.). See SACKUR (O.).

HUFF (T. H.). See HUNSAKER (J. C.).

HUNSAKER (J. C.), HUFF (T. H.), DOUGLAS (D. W.), CHOW (H. K.) and CLARK (V. E.). Dynamical stability of aeroplanes. Washington, Smithsonian Institution, 1916. 78 pp.

LANCHESTER (F. W.). Flying machines from an engineering stand-point. London, Constable, 1917. 4s. 6d.

McGegan (J. E.). Star identifier and diagrams for the graphical solution of problems in nautical astronomy. London, London Name Plate Manufacturing Company and J. D. Potter, 1916. 10s. 6d.

Maillet (E.). Cours de mécanique professé à l'école des ponts et chaussées. Paris, Hermann, 1916. 8vo.

Mellor (T.) and Pearson (H. H.). Housecraft arithmetic. (Longmans' Housecraft series for secondary schools.) London, Longmans, 1917. Cr. 8vo. 112 pp. $0.45; with answers, $0.60.

Mills (C. N.). A short course in elementary mechanics. New York, Van Nostrand, 1916 11+127 pp. $1.00

Pearson (H. H.). See Mellor (T.).

Polak (S.) and Quilter (H. C.). The teaching of drawing. Philadelphia, Lippincott, 1916. 168 pp.

Quilter (H. C.). See Polak (S.).

Riach (M. A. S.). Air screws: an introduction to the aerofoil theory of screw propulsion. London, Crosby Lockwood, 1916. 8+128 pp. 10s. 6d.

Sackur (O.). A text-book of thermochemistry and thermodynamics. Translated and revised by G. E. Gibson. London, Macmillan, 1917. 16+439 pp. 12s.

Sparagen (W.). See Wynne (W. E.).

Wynne (W. E.) and Sparagen (W.). Handbook of engineering mathematics. New York, Van Nostrand, 1917. $2.00

Book No
LIBRARY

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE thirty-ninth regular meeting of the Chicago Section, the eighth meeting of the Society at Chicago, was held at the University of Chicago on Friday and Saturday, April 6 and 7. About seventy-five persons were present, among them the following fifty-five members of the Society:

Professor G. A. Bliss, Professor R. L. Borger, Professor P. P. Boyd, Professor J. W. Bradshaw, Professor W. H. Bussey, Professor W. D. Cairns, Professor R. D. Carmichael, Dr. E. W. Chittenden, Mr. E. H. Clarke, Dr. L. C. Cox, Professor A. R. Crathorne, Mr. G. H. Cresse, Professor D. R. Curtiss, Professor L. E. Dickson, Professor A. Dresden, Professor W. B. Ford, Dr. W. V. N. Garretson, Dr. G. H. Graves, Dr. J. O. Hassler, Mr. C. M. Hebbert, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. Glenn James, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Professor W. D. MacMillan, Professor W. Marshall, Professor T. E. Mason, Dr. A. L. Miller, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Dr. A. L. Nelson, Dr. F. W. Reed, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor H. E. Slaught, Mr. G. W. Smith, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. V. H. Wells, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young, Professor A. Ziwet.

The sessions on Friday were presided over by Professor W. B. Ford, chairman of the Section; the session of Saturday forenoon was under the chairmanship of Professor L. E. Dickson, President of the Society. At the dinner held at the Quadrangle Club on Friday evening, fifty-one persons were present.

Friday afternoon was devoted to a symposium on "The Lebesgue Integral." The idea of a symposium was first suggested by Professor Van Vleck in April, 1915; it had been under discussion by different committees of the Chicago Section since that time. The committee appointed in April,

1916, for the purpose of arranging a symposium for the present meeting consisted of Professors Van Vleck and C. N. Moore, together with the members of the program committee.

The principal papers were presented by Professor Bliss and Professor Hildebrandt. Since these papers are to appear in full in later numbers of the BULLETIN, only a list of the topics treated by them will be given here.

Professor Bliss, "Integrals of Lebesgue and their applications": 1. Definition and existence of a Lebesgue integral; 2. Measure of point sets and measurable functions; 3. Properties of Lebesgue integrals; 4. Some applications of Lebesgue integrals.

Professor Hildebrandt, "Integrals, extensions of and related to the integrals of Lebesgue": 1. Integrals defined on non-measurable sets; 2. Extension of integration to non-summable functions; 3. The Stieltjes integral; 4. The Hellinger integral; 5. Generalizations of Lebesgue, Stieltjes, and Hellinger integrals.

The papers were discussed by Professor E. H. Moore and Professor E. R. Hedrick.

The forenoons of Friday and Saturday were devoted to the presentation of the following papers:

(1) Professor T. H. HILDEBRANDT: "On boundary value problems for linear differential equations in general analysis."

(2) Dr. E. W. CHITTENDEN: "On the relation of non-metrical analysis situs to the calcul fonctionnel of Fréchet."

(3) Dr. A. L. MILLER: "Projective differential geometry of three dimensional varieties in S_5 and of line complexes."

(4) Professor R. D. CARMICHAEL: "Comparison theorems for homogeneous linear differential equations of general order."

(5) Professor R. D. CARMICHAEL: "Note on convergence tests applicable to series converging conditionally."

(6) Mr. W. H. WILSON: "On a certain general class of functional equations."

(7) Dr. A. R. SCHWEITZER: "Functional equations based on iterative compositions" (second paper).

(8) Dr. A. R. SCHWEITZER: "On the implicit correlatives of certain functional equations" (preliminary communication).

(9) Mr. C. M. HEBBERT: "Circular curves generated by pencils of stelloids and their polars."

(10) Professor R. D. CARMICHAEL: "On the representation of functions in series of the form $\Sigma c_n g(x+n)/g(x)$."

(11) Dr. A. J. KEMPNER: "A theorem on lattice points."

(12) Dr. A. J. KEMPNER: "On irreducible equations admitting roots of the form $a + re^{i\theta}$, a rational and r rational."

(13) Mr. G. W. SMITH: "Nilpotent algebras generated by two generators, i and j, such that i^2 is not an independent unit."

(14) Professor J. B. SHAW: "Simplification of defining equations of nilpotent algebras."

(15) Professor E. B. STOUFFER: "On the calculation of invariants and covariants of linear homogeneous differential equations."

(16) Professor G. A. MILLER: "Substitution groups and possible arrangements of players at card tournaments."

(17) Professor T. E. MASON: "On functional solutions of certain diophantine equations" (preliminary communication).

(18) Dr. A. J. KEMPNER: "On the zeros of integral transcendental functions belonging to certain simple classes and the zeros of the derived functions."

(19) Mr. J. L. WALSH: "On the solution of linear equations in infinitely many variables by successive approximations."

(20) Dr. A. L. NELSON: "Plane nets and conjugate systems of curves."

(21) Dr. V. H. WELLS: "Single parameter systems of polar fields."

(22) Professor S. LEFSCHETZ: "On certain two-dimensional cycles belonging to an algebraic surface."

Mr. Wilson was introduced by Professor Carmichael, and Mr. Walsh by Professor Van Vleck. The papers of Professor Stouffer, Dr. Kempner, and Professor Lefschetz were read by title.

Abstracts of the papers, numbered to correspond to the titles in the list, follow below.

1. In this paper Professor Hildebrandt considers linear boundary conditions relative to the general linear differential equation of the form

$$D\eta = \alpha_0 - J\alpha\eta$$

in which η, α_0, and α are functions belonging to a class $\mathfrak{H}$, of the three variables p', p'' and x, which range over the classes $\mathfrak{P}'$ and $\mathfrak{P}''$ (general classes) and $\mathfrak{X}$ (a linear interval) re-

spectively; $D = d/dx$, and J is a linear operator on a class $\mathfrak{K}$ of functions on the range $\mathfrak{P}''\mathfrak{P}'$. The boundary conditions considered are of the form

$$c\eta(x_1) + J\sigma_1\eta(x_1) + d\eta(x_2) + J\sigma_2\eta(x_2) = \sigma_0$$

in which c and d are constants satisfying the condition $c+d=1$, σ_1, σ_2 and σ_0 are functions of the class $\mathfrak{K}$, and x_1 and x_2 are any two elements of $\mathfrak{X}$. The conditions under which there exist solutions of the differential equation and the boundary conditions are determined. By a suitable definition of adjoint boundary conditions relative to the adjoint differential equation

$$D\hat{\eta} = \alpha_0 + J\hat{\eta}\alpha$$

the usual theorems concerning the interrelations of the solutions of adjoint systems are obtained.

2. Dr. Chittenden shows for a class S, satisfying any of the sets of axioms Σ_1, Σ_2; Σ_3 given by Professor R. L. Moore as foundations for a theory of plane curves in non-metrical analysis situs, that it is possible to define the écart (P, Q) of any two elements of S, so that if $P = L_nP_n$ then $L_n(P_n, P) = 0$, and conversely. There will exist classes S_0, S_1, S_2, relative to each element P of S, such that S_0 contains P; S_2 contains S_0, S_1, and no limiting element of S_0. Hahn has shown that there exists a function $f(Q)$ which vanishes on S_0, is identically 1 on S_2, and satisfies the inequality $0 < f(Q) < 1$ on S_1. It is shown in the present paper that the solutions of the equation $f(Q) = x$, $0 < x < 1$, form in the aggregate a simple closed curve whose interior contains S_0, and therefore P.

These considerations show that non-metrical analysis situs as studied by Veblen, Lennes, and R. L. Moore is included in the calcul fonctionnel of Fréchet. A theory equivalent to the theories of these writers can therefore be based on the undefined notions point and distance. The author is continuing his investigations in this direction.

3. In this paper Dr. Miller sets up the following completely integrable set of four homogeneous linear partial differential equations in three independent variables:

$$\frac{\partial^2 y}{\partial u_1^2} = a_0^{11} y + a_1^{11} \frac{\partial y}{\partial u_1} + a_2^{11} \frac{\partial y}{\partial u_2} + a_3^{11} \frac{\partial y}{\partial u_3},$$

$$\frac{\partial^2 y}{\partial u_1 \partial u_2} = a_0^{12} y + a_1^{12} \frac{\partial y}{\partial u_1} + a_2^{12} \frac{\partial y}{\partial u_2} + a_3^{12} \frac{\partial y}{\partial u_3} + a_4^{12} \frac{\partial^2 y}{\partial u_1 \partial u_3} + a_5^{12} \frac{\partial^2 y}{\partial u_2 \partial u_3},$$

$$\frac{\partial^2 y}{\partial u_2^2} = a_0^{22} y + a_1^{22} \frac{\partial y}{\partial u_1} + a_2^{22} \frac{\partial y}{\partial u_2} + a_3^{22} \frac{\partial y}{\partial u_3},$$

$$\frac{\partial^2 y}{\partial u_3^2} = a_0^{33} y + a_1^{33} \frac{\partial y}{\partial u_1} + a_2^{33} \frac{\partial y}{\partial u_2} + a_3^{33} \frac{\partial y}{\partial u_3},$$

where the a_k^{ij}'s satisfy certain integrability conditions. The six independent solutions of these equations can be looked upon as the homogeneous coordinates of a point in five dimensions, depending on three parameters, and therefore represent a three-dimensional variety V_3 in a five-dimensional space S_5. The projective differential properties of this V_3 may be studied by means of the invariants and covariants of this set of equations under the transformations

$$\bar{y} = \lambda y \quad \text{and} \quad \bar{u}_i = \overline{U}_i(u_i).$$

A fundamental set of these invariants and covariants has been found, as well as operators that produce from this set all other invariants and covariants.

If the V_3 lies on a hyperquadric in S_5 there is a one-to-one correspondence between the points of it and the lines of a complex in S_3. Thus the differential properties of a line complex invariant under the general linear transformation on Klein coordinates may also be studied by means of this set of equations.

The only cases ruled out are those in which the V_3 lies in an S_4 or is a developable of the second kind.

4. This paper by Professor Carmichael deals with several theorems of comparison stating interesting relations between the solutions of two homogeneous linear differential equations of general order such that the coefficients of the one equation bear certain prescribed relations to those of the other equation. Among the results obtained are extensions in several directions of one of Sturm's classical theorems of comparison

for equations of the second order. (See Sturm, *Journal de Mathématiques*, 1 (1836): 106–186, and Bôcher, BULLETIN, 4 (1898): 295–313, 365–376.)

5. In this paper Professor Carmichael derives a general theorem concerning the convergence of series which need not converge absolutely and applies it to obtain some more usable particular theorems of which the following is one: If the series $a_1 + a_2 + a_3 + \cdots$ is summable of order $k - 1$ in the sense of Cesàro and if constants $c_1, c_2, c_3, \cdots$ are selected so that the series

$$\sum_{i=1}^{\infty} i^{k-1} \mid \Delta^k c_i \mid$$

converges, the limit

$$\lim_{n=\infty} n^{k-1} \Delta^{k-1} c_{n+1}$$

exists and is finite, and the quantities

$$n^{k-1} \Delta^{k-\tau} c_{n+1}, \quad (\tau = 2, 3, \cdots, k)$$

are bounded, then the series

$$a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots$$

converges (but does not necessarily converge absolutely).

6. The principal results of Mr. Wilson's paper may be stated as follows: Every solution $f(x)$ of the equation

$$(1) \qquad \sum_{i=1}^{n} \gamma_i f(\alpha_i x + \beta_i y) + \gamma_{n+1} f(x) + \gamma_{n+2} f(y) = 0,$$

in which the α's, β's and γ's are constants and no α is zero, is a solution of the normal equation

$$(2) \qquad \sum_{k=0}^{n+1} (-1)^{n+1-k} \frac{(n+1)!}{k!(n+1-k)!} f(kx + y) = 0.$$

The most general solution of (2) continuous over the finite complex x-plane is a polynomial in u and v of degree n with arbitrary coefficients, where $x = u + \sqrt{-1}\, v$, and u and v are real. The general solution of (2) analytic over the finite complex x-plane is a polynomial in x of degree n with arbitrary coefficients. The most general solution of (2) continuous

along any line not parallel to the axis of imaginaries is a polynomial in u of degree n with arbitrary coefficients, while the most general solution continuous along any line not parallel to the axis of reals is a polynomial in v of degree n with arbitrary coefficients. From these properties of the normal equation it is easy to derive the corresponding properties of equation (1). Besides these several other results are obtained incidentally.

7. Let $\{F[0, 1, 2, \cdots, n;\ i_1^{(k)} i_2^{(k)} \cdots]\}$ $(k = 1, 2, \cdots)$ be a set of iterative compositions of a function of $n + 1$ variables $(n = 1, 2, 3, \cdots)$, the variables in the entire set being notationally distinct. Then by an equation in iterative compositions Dr. Schweitzer means a relation between the preceding compositions F and some, none, or all of the variables explicitly involved. By means of suitable adjoined conditions on the variables involved special types of equations are obtained, e. g., the "equation in partial functions" of Herschel.* Generalization of the preceding to simultaneous systems is made.

An important class of equations in iterative compositions are those which involve infinite sequences and consequently the notion of convergence.† In particular, limiting cases of certain generalizations of functional equations previously defined by the author are obtained. Example: If

$$f_t^{(n)}(x) = f[t, f\{t, \cdots, f(t, x)\}],$$

where in the latter iteration the f occurs n times, and

$$\underset{n \quad \infty}{L} f[f\{f_t^{(n)}(x), x\}, f\{f_t^{(n)}(y), y\}] = f(y, x),$$

then a particular solution is

$$f(x, y) = \psi^{-1}\left\{\frac{\psi(x) - \psi(y)}{2}\right\}.$$

Finally, classes of eliminative functional equations are constructed which are satisfied by the solution of the equation $f_{t_1 t_2 \cdot t_n}^{(k)}(x) = x$, where $k = 2, 3, 4, \cdots$, and $n = 1, 2, 3, \cdots$.

* Cf. *Philosophical Transactions*, 1814, pp. 458 ff.

† Cf. the memoirs of Nicoletti, especially, *Mem. mat. fis. Soc. Ital. delle Scienze* (3), 14 (1906). The subject matter of this memoir is capable of further abstraction.

Here for $n = 2$, $k = 2$, $f_{t_1t_2}^{(2)}(x) = f\{t_1, t_2, f(t_1, t_2, x)\}$, etc. Such a class is, for instance, $f\{f_t^{(k)}(x), f_t^{(k)}(y)\} = f(x, y)$, which has the particular solution $f(x, y) = \psi^{-1}\{\xi\psi(x) + \xi\psi(y)\}$, where $\xi^{k-1} + \xi^{k-2} + \cdots + \xi^2 + \xi + 1 = 0$.

8. An important aspect of iterative compositions and functional equations is their possible expression in terms of transformations of variables and properties of these transformations, e. g., iteration of transformations. Dr. Schweitzer makes a number of applications of this theory based on equations previously defined by him. One application consists in expressing certain of his eliminative functional equations in implicit form and interpreting the solution of the latter as a problem of the inversion of the process of elimination of variables in algebra and analysis. Stated in somewhat general form, the problem is as follows: Given that an eliminant of the relations $f_i\{x_1', x_2', \cdots, x_i', x_1, x_2, \cdots, x_i, t_1, t_2, \cdots, t_k\} = 0$ $(k < i;\ i = 2, 3, \cdots)$ is $\psi\{x_1', x_2', \cdots, x_i', x_1, x_2, \cdots, x_i, f_1, f_2, \cdots, f_i, \xi\} = 0$, to find the functions f_i. Example: To determine ϕ in the equations $\phi(x_1', x_1, t) = 0$, $\phi(x_2', x_2, t) = 0$, $\phi(\xi, x_1', x_2') - \phi(\xi, x_1, x_2) = 0$, which is an implicit form of the equations $x_1' = f\{x_1, t\}$, $x_2' = f\{x_2, t\}$, $f\{x_1', x_2'\} = f\{x_1, x_2\}$. Another application consists in the solution of quasi-transitive functional equations as defined by transformations of types occurring in the theory of Lie.*

9. Mr. Hebbert studies the circular curves generated by pencils of stelloids which have the $(n + 1)$st roots of unity and their associates as base points and are connected with the transformation $z' = 1/z^n$. This is a special case of the transformation $z' = z - (n + 1)f(z)/f'(z)$, discussed by Professor Emch in the *Rendiconti del Circolo Matematico di Palermo*, volume 34 (1912), pages 1–12.

10. In this paper Professor Carmichael considers the general problem of representing functions with certain assigned properties at infinity in series of the form

$$\sum_{n=0}^{\infty} c_n \frac{g(x + n)}{g(x)},$$

* Cf., e. g., Levi-Civita, *Real. Inst. Lomb. Rendiconti* (2), vol. 28 (1895); Vessiot, same volume.

where $g(x)$ is a given function having the asymptotic representation

$$g(x) \sim x^{\mu-x}e^{\alpha+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right),$$

valid in a sector V including the positive axis of reals in its interior and being analytic in V for sufficiently large values of $|x|$. This problem contains as a special case the problem of the representation of functions in factorial series. (See *Transactions*, volume 17 (1916), pages 207–232.)

11. In this paper, Dr. Kempner discusses the system of rectilinear paths of finite width and extending to infinity in both directions which may be laid through a square lattice-point system.

Assuming in a rectangular system of coordinates the lattice-points represented by $x = 0, \pm 1, \pm 2, \cdots, y = 0, \pm 1, \pm 2, \cdots$, and considering the width d of the path as a function of the slope $\tan \varphi$, the following theorem is easily derived: When $\tan \varphi = p/q$, p, q relatively prime, then the broadest path in the direction φ which does not contain any points in its interior has the width $d = (p^2 + q^2)^{-\frac{1}{2}}$. When $\tan \varphi$ is irrational, no path of finite width exists which does not contain an infinite number of points in its interior.

Writing $d = f(\tan \varphi)$, or $y = f(x)$, it is seen that $f(x)$ is a pointwise discontinuous nullfunction which is closely related to the well known pointwise discontinuous function $y=F(x)$, $F(x) = 0$ for x irrational, $F(p/q) = 1/q$ for p, q relatively prime.

12. The present paper is a continuation of a paper: "On irreducible equations," presented by Dr. Kempner to the Southwestern Section in November, 1913. It deals with irreducible equations having among their roots some of the form $a + re^{i\theta}$, a and r both rational. Irreducibility here refers to the natural domain although extension to other domains is immediate for most of the theorems. Consider all circles in the plane, of rational radii about all real rational points as centers. One obtains thus a triply infinite system of circles which cover everywhere densely the complex plane. Some of the theorems established may then be expressed as follows: When an irreducible equation has roots on any of the circles, then

I. The equation must be of even degree;

II. There are just as many roots lying outside the circle as there are inside, and the roots outside are the points of reflection of the points inside.

III. The equation cannot have roots on any other circle which does not intersect the first; but

IV. The equation may have roots on another circle which intersects the first;

V. The radius and the center of the circle may always be determined by a finite number of rational operations and extraction of radicals;

VI. The equation is completely solvable by radicals when its degree is smaller than ten;

VII. Comparatively simple necessary and sufficient conditions for the existence of roots on any of the circles are established.

13. In this paper Mr. Smith determines the types of nilpotent algebras which are generated by two units, i and j, where i^2 is not an independent unit. He expresses these units in terms of the associative units of Professor Shaw and then forms the products ji and i^2. Since in the form

$$i, j, ij, j^2, ij^2, \cdots, j^{\mu_2-1}, ij^{\mu_2-1}, j^{\mu_2}, j^{\mu_2+1}, \cdots, j^{\mu_1-1}$$

the product of any two units is linearly expressible in terms of the units which follow both factors, certain relations among the parameters arise from the products ji and i^2. The solution of these relations leads to thirty different types of nilpotent algebras. Certain general theorems permit a simplification in the expressions for the units and the classification becomes possible.

14. In this paper Professor Shaw shows that the defining equations of a Peirce algebra (nilpotent system of one partial modulus) admit of simplifications due to a choice of units which is usually possible. The advantage of the simplification lies in the greater ease of handling the representation in associative units. If the equations in the regularized base units and the adjunct unit j are

$$ji_a = \sum_{b=1}^{m} i_b j P_{ba} + j^2 P_a \quad (a = 1, \cdots, m),$$

where P_{ba} and P_a are polynomials in j, then either the determinant

$$\begin{vmatrix} P_{11} - 1, & P_{12} & , & \cdots \\ P_{21} & , & P_{22} - 1, & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

vanishes or does not vanish. In the latter case the base units i may be so chosen as to reduce P_{ba} to j merely for $b = 1, \cdots, m$. In the former case when the second term in the principal diagonal elements is replaced by λ, the determinant set equal to 0 has m roots, which are polynomials in j. If these are all distinct the terms in j^2P_a all drop out for a properly chosen base. If the roots are not distinct the problem needs further study.

15. In previous papers presented to the Society, Professor Stouffer has obtained complete systems of seminvariants and semicovariants for a system of ordinary linear homogeneous differential equations of the second order with n dependent variables. In the present paper the problem is completed by the calculation of complete systems of invariants and covariants for this system of differential equations. The more complicated invariants and covariants are obtained by applying certain operators to a set of simple invariants and covariants.

16. To emphasize certain properties of substitution groups Professor Miller directs attention to their availability for the purpose of determining arrangements of the players at card tournaments so that during a series of games each player will have each of the others once as a partner and twice as an opponent. The possibility of determining such arrangements was considered, from a different point of view, by E. H. Moore in his "Tactical memoranda" published in volume 18 of the *American Journal of Mathematics*.

When the number of players is a power of 2, sets of possible arrangements satisfying the stated condition can be obtained directly by means of the regular substitution group of order 2^m, and of type $(1, 1, 1, \cdots)$, by starting with any subgroup of order 4 contained in such a substitution group and transforming this subgroup under the powers of an operator of order $2^m - 1$ in the group of isomorphisms of this substitution group. When m is even we may impose the additional condition that during the first $(2^m - 1)/3$ games each of the

players has each of the others once and only once either as a partner or as an opponent, but sets in which this additional condition is not satisfied can be obtained with equal ease in this case.

17. In this paper Professor Mason replaces, in certain equations of Diophantus, the unknowns which are to be rational numbers by unknown functions restricted only by the equations. The functional equations are then solved. These functional equations, as such, are of some interest. The problem remaining is to restrict these functional solutions so that they will satisfy the conditions of Diophantus. The number of diophantine equations to which this process may be applied seems large.

18. In the first part of this paper Dr. Kempner unifies a set of known theorems (comprising, for example, the first two theorems of § 230, and the theorems of §§ 232, 233, 234 in Vivanti-Gutzmer, Analytische Funktionen, 1906); their proofs are slightly simplified by introducing convex polygons (open or closed) which extend to infinity in the complex plane. As an application, the following theorem is derived:

Let $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $c_0 \neq 0$, converge in the whole complex plane, let a_ν, $(\nu = 1, 2, \cdots)$, be the zeros of $f(z)$, and let $\Sigma |a_\nu|^{-\epsilon}$ be convergent for every $\epsilon > 0$; if any one of the $c_1, c_2, \cdots$ is zero, then either the zeros of $f(z)$ lie all on a straight line passing through the origin, or every straight line through the origin has zeros on either side. (All a and c are real or complex.) In the second part theorems of the following kind are derived: Let m, n be any two positive integers, and $a_1, \cdots, a_n$ any n numbers, $b_1, \cdots, b_m$ any m numbers, real or complex, and arbitrary except that no two a's shall be equal, and no a equal to any b. Then a function $f(z)$ exists such that

I. $f(z) = e^{g(z)} \cdot \varphi(z)$, where $g(z)$ and $\varphi(z)$ are polynomials;

II. $f(z) = 0$ has exactly the roots $a_1, \cdots, a_n$;

III. $f'(z) = 0$ has exactly the following roots: (a) $b_1, \cdots, b_m$ (all single, or counted with proper multiplicity when $b_1, \cdots, b_m$ are not all distinct), (b) up to $n - 1$ other roots, which cannot be arbitrarily assigned and which in general depend both on the given a and b.

The author does not believe that this theorem can be derived by specialization from a (possibly related) theorem of much

wider scope due to Guichard (*Annales de l'Ecole Normale Supérieure*, 1884).

19. Mr. Walsh considers a system of equations

$$x_i + \sum_{j=1}^{\infty} a_{ij} x_j = c_i \quad (i = 1, 2, 3, \cdots)$$

under the restrictions

$$\sum_{j=1}^{\infty} |a_{ij}| \leqq p < 1, \quad |c_i| \leqq C.$$

A solution is obtained by using successive approximations; this solution is such that the x's are bounded, and it is the only such solution.

By a slight change in the form of the equations, the results are extended to include some well known results of Poincaré and von Koch.

20. G. Koenigs has proved that the perspectives on a fixed plane from a fixed point of the asymptotic curves of a surface form a net with equal Laplace-Darboux invariants. Dr. Nelson's paper deals with the perspectives of conjugate systems of curves on a surface. Use is made of Wilczynski's methods, according to which the conjugate system of curves is considered as being generated by the point P_y whose homogeneous coordinates, $y^{(k)} = y^{(k)}(u, v)$ $(k = 1, 2, 3, 4)$, constitute a set of linearly independent solutions of the completely integrable system of equations

$$y_{uu} = ay_{vv} + by_u + cy_v + dy,$$
$$y_{uv} = b'y_u + c'y_v + d'y.$$

A point, P_τ, which describes a plane net of curves, has the homogeneous coordinates $\tau^{(k)} = \tau^{(k)}(u, v)$ $(k = 1, 2, 3)$, which are linearly independent solutions of the completely integrable system

$$(T) \qquad \begin{aligned} \tau_{uu} &= A\tau_u + B\tau_v + C\tau, \\ \tau_{uv} &= A'\tau_u + B'\tau_v + C'\tau, \\ \tau_{vv} &= A''\tau_u + B''\tau_v + C''\tau. \end{aligned}$$

The following theorems are obtained. (*a*) Any plane net

may be considered as the perspective from a fixed point of a conjugate system of surface curves. (*b*) The perspective on a fixed plane from a fixed point of a given conjugate system of surface curves forms a plane net whose completely integrable system (T) is characterized by the mixed invariantive relation

$$\frac{\partial^2}{\partial u \partial v} \log a - \frac{\partial}{\partial v}(aA'') + \frac{\partial}{\partial u}\left(\frac{B}{a}\right) - \frac{\partial A'}{\partial u} + \frac{\partial B}{\partial v} = 0.$$

21. The paper of Dr. Wells is a study of the five single parameter systems of polar fields, the pencil, the range, and the three intervening mixed systems. The pencil of polar fields is defined as all the polar fields determined by triangles perspective and reciprocal to one triangle, such that the corresponding sides of those triangles form three projective first order pencils of rays. Similar definitions are given for the other forms by means of the determining triangles. In the pencil of polar fields the centers of perspective for the determining triangles form a range of the fourth order, and the lines of perspective a pencil of the second order. For each system, the principal theorems give the loci of the poles of a line, and the polars of a point. In the first mixed system the polars of a point form a second order pencil, and the poles of a line a fourth order range.

22. In this note Professor Lefschetz establishes the existence of a two dimensional cycle Γ, corresponding to any two algebraic curves C_1, C_2 on an algebraic surface $F(x, y, z) = 0$. By means of certain functions of Poincaré, the periods of a double integral

$$(1) \qquad \int\int \frac{P(x, y, z)}{F_z'} dxdy \quad (P \text{ adjoint polynomial})$$

are found. From these it follows that if $t_1C_1 = t_2C_2$, $\Gamma \sim 0$. If (1) is of the form

$$\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dxdy,$$

the periods are linear combinations of the logarithmic periods of

$$\int V(x, \bar{y}, z)dx; \quad F(x, \bar{y}, z) = 0,$$

from which a very interesting connection with Severi's theory of the base is derived. The paper appeared in the *Rendiconti dei Lincei* for February.

ARNOLD DRESDEN,
Secretary of the Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-ninth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 7. Two sessions were required for the presentation of the program. Professor Lehmer occupied the chair. The following members of the Society were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer, Dr. H. N. Wright.

It was voted to hold the next meeting of the Section at the University of California, on October 27.

The following papers were read at this meeting:

(1) Professor H. F. BLICHFELDT: "A further reduction of the known maximum limit to the least value of quadratic forms."

(2) Professor D. N. LEHMER: "Certain divisibility theorems concerning the convergents of Hurwitzian continued fractions."

(3) Dr. G. F. MCEWEN: "Determination of the functional relation between one variable and each of a number of correlated variables by successive approximation."

(4) Professor W. A. MANNING: "On the order of primitive groups (III)."

(5) Dr. H. N. WRIGHT: "Note on a certain quadratic transformation of the plane."

In the absence of Dr. McEwen, his paper was read by Mr. E. F. Michael of the Scripps Institute for Biological Research. Abstracts of the papers follow below.

1. Having given the determinant D of a positive-definite quadratic form F in n variables, such integers, not all zero,

can be assigned to the variables that the value of F is not greater than $\gamma_n D^{1/n}$, where γ_n is a number depending only upon n (cf. *Transactions* for 1914, pages 227 ff.). Professor Blichfeldt has recently obtained a lower value for γ_n than that hitherto known (l. c., page 233) when $n > 5$.

2. Hurwitz (*Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 1896) has studied continued fractions whose partial quotients belong to arithmetical series. Taken modulo n the partial quotients of such fractions recur, and so also do the convergents. The number of terms in the period of the convergents seems to depend in a very simple way on the modulus, and the appearance of prime factors in the numerators and denominators of the convergents presents an interesting study. The continued fraction for the Naperian base $e = (2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, \cdots)$ is an example. Calling the kth convergent A_k/B_k, Professor Lehmer has shown that A_k recurs with a period of $3n$ with respect to any modulus n, and B_k has a period $6n$. Also $B_{3n-2} \equiv B_{3n-6} \equiv A_{3n-3} \equiv 0 \pmod{n}$.

3. In order to determine quantitative relations from "field" data, where all the related magnitudes vary simultaneously, the disturbing effect due to the interrelationship of the independent variables must be eliminated. Assuming the change in the dependent variable w, due to a given change in any one of the independent variables x, y, z, etc., to be independent of the values of the other variables, the problem is to determine the functional relation of w to each variable, when the form of the function is unknown and unrestricted.

Arrange the values of x in order of magnitude, and enter corresponding values of w. Divide this series of values of x into groups, each containing about the same number of entries and having about the same range, and find the regression of w on x in each group, assuming it to be linear in each. Proceed in the same way with each of the other independent variables, thus obtaining first approximations to the average value of w and to each regression coefficient in each group corresponding to the average value of the independent variable from which the group was formed. By means of these approximate relations of w to y, w to z, etc., apply a correction to each entry in the (w, x) tabulation that will reduce the value of w

to what it would have approximately been if y, z, etc., had constant values, arbitrarily chosen. Then redetermine the regression of w on x in each group, and proceed in the same way with the other independent variables.

A set of determinate simultaneous linear equations can be derived whose roots are the true regression coefficients and averages required, and it can be shown that if the values of the averages and regression coefficients found by continuing the above approximation process converge, the limits will satisfy these equations.

Formulas, readily derived and involving constants easily determined from the data, greatly facilitate the computation, and provide an absolute check on the numerical work of each approximation, after the second.

Dr. McEwen will offer this paper to the *Annals of Mathematics.*

4. A partial statement of Professor Manning's theorem is this: Let q be any positive integer greater than unity, and let p be a prime number greater than $2q$; then the degree of a primitive group that contains a substitution of order p and degree qp but none of order p and of degree less than qp cannot exceed $qp + 4q - 4$. This paper has been offered to the *Transactions.*

5. An involution of lines is set up about each of the points A and B such that the double lines of each involution are at right angles. Then any point P of the plane, as the intersection of a line of A with a line of B, corresponds to a point P' as the intersection of the corresponding lines of A and B. And conversely P' is seen to correspond to P. In this way is obtained a quadratic transformation of the plane.

The involutions about A and B determine a third involution of the same character about a point C, such that the same quadratic transformation of the plane is obtained from any two of the three involutions.

Dr. Wright shows that in this transformation the ideal line of the plane goes into the circle through A, B, and C. This circle is found to be the nine-point circle of any triangle whose vertices are three of the four invariant points of the transformation.

W. A. MANNING,
Secretary of the Section.

SINGULAR POINTS OF ANALYTIC TRANSFORMATIONS.

BY PROFESSOR WILLIAM F. OSGOOD.

LET a transformation be defined by the equations

$$x_i = \varphi_i(u_1, \ldots, u_n) \qquad (i=1, \ldots, n),$$

where the functions φ_i are meromorphic in the origin and the first μ of them, $0 < \mu \leqq n$, have a non-essential singularity of the second kind there, the remaining $n - \mu$ functions being analytic or having a pole there. Let the point (x) be interpreted in the space of analysis.

Let those points of the region

$$|u_k| < \eta_k \qquad (k = 1, \ldots, n),$$

where η_k is a positive number, in which no function φ_i has a non-essential singularity of the second kind, be denoted by $\mathfrak{T}$, and let the points (x) which form the images of the points (u) of $\mathfrak{T}$ constitute the region M. There will be certain points of the (x)-space which will lie on the boundary of M, no matter how far $\mathfrak{T}$ is restricted. The manifold of these points shall be denoted by $\mathfrak{M}$.

The object of this note is to communicate the following theorem, the proof of which will shortly be published elsewhere.

THEOREM. *The manifold $\mathfrak{M}$ is made up of a finite number of algebraic manifolds of the following kind:*

In the space of the first μ variables $(x_1, \ldots, x_\mu)$ there exists a manifold $\mathfrak{K}$ formed by a finite number of irreducible algebraic curves $(k = 1)$, surfaces $(k = 2)$, or hypersurfaces of order $k < \mu$, the number k being the same for all; or finally, if $\mu < n$, $\mathfrak{K}$ may include all the points of the space in question, and we set here $k = \mu$.

Then $\mathfrak{M}$ consists of the points $(x_1, \ldots, x_n)$, where $(x_1, \ldots, x_\mu)$ is an arbitrary point of $\mathfrak{K}$, and

$$x_j = a_j = \frac{g_j(0, \ldots, 0)}{G_j(0, \ldots, 0)} \qquad (j = \mu + 1, \ldots, n).$$

HARVARD UNIVERSITY,
April, 1917.

THE PROJECTION OF A LINE SECTION UPON THE RATIONAL PLANE CUBIC CURVE.

BY PROFESSOR JOSEPH EUGENE ROWE.

(Read before the American Mathematical Society, April 28, 1917.)

Introduction.

THE rational plane curve of the third order, which we shall refer to as the R^3, is of the fourth class; that is, from an arbitrary point of the plane four tangents can be drawn to the curve. But if the point is selected on the R^3 itself, the tangent at the point accounts for two of these tangents, and, therefore, from such a point only two additional tangents can be drawn to the curve. A line section yields three points of the R^3 and these, in the manner just described, determine three pairs of additional tangents. An investigation of the points of a line and the six tangents so determined shows that the relations which exist among these are interesting as well as of a fundamental character.

We shall let

$$x_i = a_i t^3 + 3b_i t^2 + 3c_i t + d_i. \qquad (i = 0, 1, 2) \tag{1}$$

be the parametric equations of the points of the R^3, and it has been found convenient to use the following abbreviations:

$$\alpha = |\, abc \,|, \quad \beta = |\, abd \,|, \quad \beta' = |\, acd \,|, \quad \alpha' = |\, bcd \,|. \tag{2}$$

Also, it may be verified that the identities

$$a_i\alpha' - b_i\beta' + c_i\beta - d_i\alpha = 0 \tag{3}$$

exist among the coefficients in (1) and the Greek letters of (2).

The Choice of a Line Section.

As the parameters 0 and ∞ may be assigned to any two elements in a one-dimensional space, we select the line determined by the points of the R^3 whose parameters are 0 and ∞. From (1) it follows that the coordinates of these points are d_i and a_i, respectively; hence the equation of the line determined by them is $|\, adx \,| = 0$, and the parameter of the third point

of the R^3 (found by substituting from equations (1) in $|adx| = 0$) collinear with a_i and d_i is the root of

$$\beta t + \beta' = 0. \tag{4}$$

By substituting $t = -\beta/\beta'$ in (1) we obtain for the coordinates of the point (4)

$$x_i = -a_i\beta'^3 + 3b_i\beta'^2\beta - 3c_i\beta'\beta^2 + d_i\beta^3 \quad (i=0, 1, 2). \tag{5}$$

The Projections of the Three Points upon the R^3.

The projection of a point x_i upon (1) is*

$$|abx|t^4 + 2|acx|t^3 + (|adx| + 3|bcx|)t^2 + 2|bdx|t + |cdx| = 0. \tag{6}$$

That is, if the coordinates of a point x_i are substituted in (6), the result is a quartic in t whose roots are the parameters of the points of contact of the four tangents that can be drawn from x_i to the R^3 of (1).

By substituting a_i, d_i, and the coordinates (5) in (6) for x_i we obtain

$$3\alpha t^2 + 2\beta t + \beta' = 0, \tag{7}$$

$$\beta t^2 + 2\beta' t + 3\alpha' = 0, \tag{8}$$

$$(\beta^2 - 3\alpha\beta')t^2 + (3\alpha'\beta - \beta'^2) = 0, \tag{9}$$

whose roots are the parameters of the points of contact of the tangents to R^3 drawn from the points whose parameters are ∞, 0, and $-\beta'/\beta$, respectively.

The form of equations (7)–(9), if properly interpreted, conveys a great deal of information. Evidently the roots of (9)† are harmonic‡ (apolar) to 0 and ∞; the roots of (8) are harmonic to ∞ and $-\beta'/\beta$; and the roots of (7) are harmonic to 0 and $-\beta'/\beta$. These results we summarize in

THEOREM I. *The parameters of any two of three collinear points on the R^3 are harmonic to the parameters of the points of contact of the two additional tangents that can be drawn to the R^3 from the third point.*

* J. E. Rowe, BULLETIN, vol. 22, No. 2, p. 75 (November, 1915).

† Observe that the order of statement in this sentence is not without purpose.

‡ Salmon, Higher Algebra, fourth edition, p. 179.

Also, the determinant of equations (7)–(9) vanishes, as may be seen at once from the fact that (9) may be obtained by subtracting β' times (7) from β times (8). Hence* we have

THEOREM II. *The parameters of the points of contact of the three pairs of tangents that can be drawn to the R^3 from three collinear points of the R^3 are harmonic to the same quadratic, or form a set in involution.*

Another result which may be derived as a corollary of Theorem I we shall state as

THEOREM III. *Lines on a point P of an R^3 cut the R^3 in pairs of residual points whose parameters are harmonic to the parameters of the points of contact of the two additional tangents drawn to R^3 from P.*

Although Theorem III may be regarded a corollary of Theorem I, it may be established independently. Thus: Let $P(d_0, d_1, d_2)$ be the point and $(\kappa x) \equiv \kappa_0 x_0 + \kappa_1 x_1 + \kappa_2 x_2 = 0$ any line on P. Then $(\kappa d) = 0$. The parameters of the residual points cut out of (1) by $(\kappa x) = 0$ are the roots of

$$(\kappa a)t^2 + 3(\kappa b)t + 3(\kappa c) = 0 \tag{10}$$

and (10) is apolar to (8), for

$$3(\kappa c)\beta + 3(\kappa a)\alpha' - 3(\kappa b)\beta' = 0,$$

as may be shown from (3) and the fact that $(\kappa d) = 0$.

PENNSYLVANIA STATE COLLEGE,
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EXAMPLES OF A REMARKABLE CLASS OF SERIES.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, April 28, 1917.)

Two-Fold and One-Fold Expression of the Properties of Functions.

1. IN the development of analysis during the past generation it has frequently happened that functions have arisen which are analytic in a sector of the complex plane and in

* Salmon, Higher Algebra, fourth edition, p. 180.

that sector have a determinate asymptotic representation (in the sense of Poincaré) by means of a diverging power series. The first explicit appearance of these functions in a general way was in two memoirs by Poincaré* in 1885 and 1886.

The importance of these functions was at once put in evidence by Poincaré's association of them with the irregular singular point at infinity of the linear homogeneous differential equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0,$$

in which the coefficients $P_0, P_1, \cdots, P_n$ are polynomials in x subject to the condition that their degrees do not constantly decrease as one passes in order from the first to the last polynomial in the sequence. Poincaré (*Acta Mathematica*, loc. cit.) has in fact shown that a solution $y(x)$ of such an equation is in general asymptotic to an expression of the form

$$x^{\mu} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right),$$

where μ is a constant and $Q(x)$ is a polynomial in x, the asymptotic representation being valid for x approaching infinity in an appropriate direction.

In other important investigations functions of the same character have often made their appearance, particularly in the theory of differential and difference equations. In these investigations it has been usual to represent the asymptotic character of such a function by means of a diverging power series and to find in addition another form through which to put in evidence the analytic character of the function in the sector. In this connection use has frequently been made of the integral on which the Laplace transformation is based or of certain generalizations of it. In other cases it has been found necessary to resort to expansions of various types, particularly of those which arise through use of the method of successive approximation. These procedures do not always lead to suitably convenient representations of the functions in consideration; in particular, the expansions last mentioned are often complicated in character.

* *Amer. Jour. of Math.*, vol. 7 (1885), pp. 203–258; *Acta Mathematica*, vol. 8 (1886), pp. 295–344. Particular instances of such functions appeared earlier, notably in the work of Stirling and Cauchy (see Borel's Séries divergentes, p. 30).

2. Obviously, it is desirable, if possible, to have a single expansion of such character that it is capable of exhibiting the asymptotic properties of the function near infinity and of yielding at the same time a convenient and workable representation of it in the finite plane or in a significant portion of the finite plane. It is hardly to be expected that any single class of series will afford a form of such a tool best suited to all situations; but it may very well be (and we are about to exhibit a class of series in which it is) true that certain classes of functions of great importance and very frequently recurring in practice are capable of representation in one or another sort of series all of which belong to a single type and possess a unitary theory.

Indeed it has recently appeared that factorial series and certain immediate generalizations of them serve just such a purpose to a remarkable degree. Nörlund* has shown that functions which are characterized by being analytic in a half-plane and having an asymptotic character of a certain broad type are expansible in convergent factorial series (of a slightly generalized form) and that these series do in fact readily show the asymptotic character of the functions represented by them. In connection with a range of ideas differing in some essential respects from those of Nörlund it was shown earlier by Watson† that a certain form of factorial series furnished a sufficient means of representing a very important and wide-reaching class of functions. About the same time Horn‡ employed another generalization of factorial series in obtaining suitably convenient representations of solutions of linear differential equations in the neighborhood of an irregular singular point. Nörlund§ has likewise employed factorial series in his fundamental investigation of the solutions of difference equations.

3. In a recent memoir|| I have pointed out that factorial series whose fundamental importance has been made manifest in these several ways are but an instance of a large class of series of simple properties. It turns out that the latter are

* *Acta Mathematica*, vol. 37 (1914), pp. 327–387.

† *Rendiconti del Circolo Matematico di Palermo*, vol. 34 (1912), pp. 41–88.

‡ *Math. Annalen*, vol. 71 (1912), pp. 510–532.

§ *Rendiconti del Circolo Matematico di Palermo*, vol. 35 (1913), pp. 177–216.

|| *Transactions Amer. Math. Soc.*, vol. 17 (1916), pp. 207–232. See also a forthcoming paper in the *Amer. Jour. of Math.*

suitable for the representation of functions which are defined throughout a sector and have certain types of singularities at infinity. Moreover these series afford the requisite information concerning the asymptotic character of the functions represented by them.

These series are of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where $g(x)$ is a given function of x and the coefficients c_n are independent of x. More precise definitions are to be found in the next sections. As will be seen from these definitions, the fundamental characteristic of $g(x)$ is its asymptotic behavior.

A detailed study of the more important series included in this class and especially of their suitability for the representation of certain types of functions is desirable and leads to important consequences in the theory of functions. From these latter the close connection of the series $\Omega(x)$ with the function-theoretic considerations mentioned in the preceding sections will become apparent.

The principal object of the present paper is to show how several important series in the literature are included as special cases of the series $\Omega(x)$ and the generalization $T(x)$ of it introduced in section 6 and to exhibit certain other interesting examples. I hope that attention may thus be directed to the importance of this entire class of series. I add also a brief discussion of the region of convergence of the new series $T(x)$.

Definition of the Series $\Omega(x)$ and $T(x)$.

4. The class of series $\Omega(x)$ is determined primarily by the properties of the *basic function* $g(x)$ through which the series are defined. Accordingly it is desirable to begin with a statement of the properties of suitable functions $g(x)$.

In every case we shall confine our attention to a defined portion of the complex plane and usually to a sector formed by two rays from zero to infinity and containing in its interior the positive axis of reals. In such a portion of the plane (appropriately defined) we assume that $g(x)$ is single-valued and also that it is analytic when $|x|$ is sufficiently large. Moreover it is to have the asymptotic properties expressed in the formula

$$(1) \qquad g(x) \sim x^{P(x)}e^{Q(x)}\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

in which $P(x)$ and $Q(x)$ are polynomials which we write in the form

$$P(x) = \mu_0 + \mu_1 x + \mu_2 x^2 + \cdots + \mu_k x^k \quad (\mu_k \neq 0 \text{ if } k > 0),$$

$$Q(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_m x^m \quad (\alpha_m \neq 0 \text{ if } m > 0).$$

By r^t we mean $e^{t \log r}$, where the principal determination of $\log r$ is taken.

In case $k = 0$ we assume that $m > 1$ in order to avoid a case which is unimportant so far as our present objects are concerned.

The asymptotic relation (1) is to be understood as an abbreviation for the infinite sequence of limits

$$(2) \quad \lim x^s\left(g(x)x^{-P(x)}e^{-Q(x)} - \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_s}{x^s}\right)\right) = 0 \quad (s = 0, 1, 2, \cdots),$$

the limits being taken for x approaching infinity in a sector including the positive axis of reals in its interior.

5. By means of any function $g(x)$ possessing the properties just indicated we introduce the series $\Omega(x)$ defined by the relation

$$(3) \qquad \Omega(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where the coefficients $c_0, c_1, c_2, \cdots$ are independent of x.

In case $g(x+n)$ and $g(x)$ have common zeros we shall understand that $g(x+n)/g(x)$ denotes the function obtained when the numerator and denominator of this fraction are divided by an entire function (preferably a polynomial when possible) having for its zeros the common zeros of $g(x+n)$ and $g(x)$.

A value of x for which any one of the functions $g(x)$, $g(x+1)/g(x)$, $g(x+2)/g(x)$, $\cdots$ has a singularity will be called an *exceptional value* or an *exceptional point* for the series $\Omega(x)$. Other values of x will be called *non-exceptional* or *ordinary*.

In order to avoid cases without value for our purposes we shall assume that $g(x)$ is of such character that there is a

(two-dimensional) region in the x-plane containing no exceptional points for the series.

6. Let us suppose that we have a finite number $\nu + 1$ of series $\Omega(x)$ which, with x replaced by x^t where t is a positive integer, we write in the form

$$\Omega_i(x^t) = \sum_{n=0}^{\infty} c_{ni} \frac{g(x^t + n)}{g(x^t)} \quad (i = 0, 1, 2, \cdots, \nu).$$

Let us multiply $\Omega_i(x^t)$ by $1/x^i$ $(i = 0, 1, 2, \cdots, \nu)$ and add the resulting series term by term. We are thus led to a new series

$$T_{\nu t}(x) = \sum_{n=0}^{\infty} P_{n\nu}(x) \frac{g(x^t + n)}{g(x^t)}, \tag{4}$$

where $P_{n\nu}(x)$ is a polynomial of degree ν in $1/x$, namely,

$$P_{n\nu}(x) = c_{n0} + \frac{c_{n1}}{x} + \frac{c_{n2}}{x^2} + \cdots + \frac{c_{n\nu}}{x^\nu}.$$

Exceptional points and ordinary points are here defined as in section 5, the point $x = 0$ being now added to the set of exceptional points.

These series $T(x)$ are here introduced for the first time. It is clear that they have a much greater flexibility than series $\Omega(x)$ in adapting themselves to the necessities of representing a function which is arbitrary save as to general properties.

7. The definitions of series $\Omega(x)$ and $T(x)$ might be extended to the case in which (1) is replaced by the relation

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x^{1/p}} + \frac{a_2}{x^{2/p}} + \cdots\right)$$

and $P(x)$ and $Q(x)$ are polynomials either in x or more generally in $x^{1/p}$. Suitable functions $g(x)$ for use in defining these more general series arise in situations analogous to those mentioned in sections 8 and 9 below. The theory of these more general series $\Omega(x)$ and $T(x)$ will not be treated in this paper.

Convenient Sources of Suitable Functions $g(x)$.

8. It is obvious that there exists a large class of series $\Omega(x)$ in accordance with the foregoing definition; for one has the subclass of functions $g(x)$ for which the series in the second

member of (1) is convergent when $|x|$ is sufficiently large and in particular that in which the series terminates, the case in which it consists of the first term alone being of especial interest (as we shall see in section 11).

That there also exist other important basic functions $g(x)$ may be readily seen. In fact, as we saw in section 1, solutions $y(x)$ of linear homogeneous differential equations of a certain broad type are found to possess in general the asymptotic representation

$$y(x) \sim x^{\mu} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right)$$

valid for x approaching infinity in an appropriate direction, $Q(x)$ being a polynomial in x and $\mu, a_1, a_2, \cdots$ being constants. Hence a constant c exists such that $y(cx)$ is a function having the property specified above for $g(x)$.

9. Another class of functions $g(x)$ is afforded by the theory of difference equations. These functions are usually more effective than those defined by differential equations because the sequence of functions

$$\frac{g(x+1)}{g(x)}, \quad \frac{g(x+2)}{g(x)}, \quad \frac{g(x+3)}{g(x)}, \quad \cdots$$

has in this case very simple properties of interrelation among its elements, more convenient for most purposes than in the case of functions defined by differential equations. A theorem asserting the existence of these functions may be stated as follows:*

Let us consider the linear homogeneous difference equation of order n

$$F(x+n) + a_1(x)F(x+n-1) + \cdots + a_n(x)F(x) = 0,$$

in which the coefficients $a_k(x)$ have the property that $x^{-\mu k} a_k(x)$ is analytic at infinity for every value of k from 1 to n, μ being a constant. By $e^{\mu k} a_{k0}$ we denote the value of $x^{-\mu k} a_k(x)$ at infinity. We assume that the quantities a_{k0} are such that the characteristic algebraic equation

$$\alpha^n + a_{10}\alpha^{n-1} + \cdots + a_{n0} = 0$$

has its roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ different from each other and from

* See *Amer. Jour. of Math.*, vol. 38 (1916), pp. 185–220, and the papers there referred to.

zero. In this case the difference equation has n formal (and in general divergent) power series solutions of the form

$$F_i(x) = x^{\mu x}\alpha_i^x x^{\mu_i}\left(1 + \frac{c_{i1}}{x} + \frac{c_{i2}}{x^2} + \cdots\right) \quad (i = 1, 2, \cdots, n),$$

where $\mu_i, c_{i1}, c_{i2}, \cdots$ are constants which may be reckoned out directly by substituting the expansion in the difference equation and equating like powers of x in the result.

Then the difference equation has a fundamental system of solutions

$$F_1(x),\ F_2(x),\ \cdots,\ F_n(x)$$

possessing the following properties:

(1) Each function in the system is analytic throughout the finite plane except at the singularities α of the functions

$$\frac{1}{a_n(x)},\quad \frac{a_1(x)}{a_n(x)},\quad \frac{a_2(x)}{a_n(x)},\quad \cdots,\quad \frac{a_{n-1}(x)}{a_n(x)}$$

and the points $\alpha - i$ where i is a positive integer.

(2) The functions $F_1(x), F_2(x), \cdots, F_n(x)$ have with respect to x the asymptotic representation

$$(5)\quad F_i(x) \sim x^{\mu x}\alpha_i^x x^{\mu_i}\left(1 + \frac{c_{i1}}{x} + \frac{c_{i2}}{x^2} + \cdots\right) \quad (i = 1, 2, \cdots, n),$$

this representation being valid for x approaching infinity in any sector V formed by two rays from zero to infinity with arguments less than $\pi/2$ in absolute value.

Very useful particular basic functions $g(x)$ for series $\Omega(x)$ and $T(x)$ are afforded directly by the foregoing functions $F_i(x)$, especially in the case when μ is a negative integer. If μ is not a negative integer then a suitable and convenient function $g(x)$ in place of $F_i(x)$ is given by

$$g(x) = F_i(x)\ [\Gamma(x)]^{-\mu-k},$$

where k is a positive integer and $\Gamma(x)$ denotes the gamma function of Euler. For the case when k is unity these functions are especially useful. Moreover, any one of many expressions similar to the foregoing readily serves the same purpose.

The case when the difference equation is of the first order

deserves special mention on account of the elegance of many of the series arising through functions $g(x)$ satisfying equations of the form

$$g(x+1) = R(x)g(x),$$

where $R(x)$ is a rational function of x. See sections 12–17 below.

10. Series $\Omega(x)$ and $T(x)$ give rise to what appears to be the fundamental expansion problem in the theory of difference equations; or, more exactly, they give rise to a general problem which includes this one as a special case and also a corresponding one in the theory of differential equations. The general problem is far more comprehensive than either of these special cases.

The total class of series $\Omega(x)$ and $T(x)$ is very large owing to the paucity of restrictions which it is necessary to put upon the basic functions $g(x)$. So far as I can see now, however, it appears probable that the functions $g(x)$ afforded by the solutions of linear homogeneous difference equations give rise to the most flexible series in the total class and are thus likely to prove themselves of the greatest permanent use in developing the theory of functions. It is for this reason that I venture to suggest tentatively that the series treated in this paper give rise to the fundamental expansion problem in the theory of difference equations.

It is clear, however, that this expansion problem is not analogous to that having to do with orthogonal and biorthogonal functions in the theory of differential equations. Owing to the fact that many common properties are possessed by difference and differential equations it is natural to expect in the theory of the former the analogue of expansions in fundamental functions arising in the theory of the latter. That such a theory could be developed is highly probable; but it is not at present in existence. It seems improbable, however, that the expansions so obtainable are commensurate in importance with expansions in the form of our series $\Omega(x)$ and $T(x)$.

Special Cases of Series $\Omega(x)$ and $T(x)$ in the Literature.

11. Let us consider the special case in which $g(x)$ has the value

$$g(x) = e^{\sigma x^2}.$$

Then $\Omega(x)$ is readily reduced to the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(n) e^{2\sigma n x}.$$

If we make the substitution $z = e^{2\sigma x}$ the series $\Omega(x)$ is transformed into the power series

$$\sum_{n=0}^{\infty} c_n g(n) z^n.$$

Hence among the special cases of series $\Omega(x)$ is to be found the foregoing elementary transform of a general power series.

12. It is well known that the function $\Gamma(x)$ has the asymptotic character

$$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \cdots \right).$$

Hence for $g(x)$ we may take the special value

$$g(x) = \frac{1}{\Gamma(x)} \sim x^{-x+\frac{1}{2}} e^{x} \left(a_0 + \frac{a_1}{x} + \cdots \right).$$

Then we have

$$\frac{g(x+n)}{g(x)} = \frac{\Gamma(x)}{\Gamma(x+n)} = \frac{1}{x(x+1)(x+2)\cdots(x+n-1)}.$$

Hence for this case our Ω-series takes the form

$$\Omega(x) = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{x(x+1)(x+2)\cdots(x+n-1)}.$$

This is the well-known *factorial* series. The so-called *series of binomial coefficients* arises similarly by taking for $g(x)$ the value

$$g(x) = \Gamma(x),$$

whence we have

$$\Omega(x) = c_0 + \sum_{n=1}^{\infty} c_n x(x+1)\cdots(x+n-1).$$

13. Another interesting example of our series $\Omega(x)$ is obtained by taking for $g(x)$ the value

$$g(x) = \frac{1}{a^x \Gamma(x + b/a)} \sim x^{-x+\mu} e^{\alpha+\beta x} \left(1 + \frac{a_1}{x} + \cdots \right),$$

where a and b are constants and $a \neq 0$. In this case the Ω-series takes the form

$$\Omega(x) = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{(ax+b)(ax+b+a)\cdots(ax+b+(n-1)a)}.$$

Let us now give to b the value 0 and replace ax by z. Then if we put $c_0 = 0$ the foregoing series takes the form

$$(6) \qquad S(z) = \sum_{n=1}^{\infty} \frac{c_n}{z(z+a)(z+2a)\cdots(z+(n-1)a)}.$$

This series plays the leading rôle in Nörlund's fundamental paper on factorial series referred to in section 2.

Concerning series $S(z)$ Nörlund proposes a fundamental problem in the following manner: What is the class of functions which admit an expansion of the form $S(z)$ for appropriately chosen real values of a? To this question he finds a very interesting answer. He shows that the class of functions is the same as that which gives rise to power series of the form

$$\frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \cdots,$$

which may be divergent but which are absolutely and uniformly summable by the exponential method of Borel. An infinite number of functions may give rise to the same power series; but it is the function which Borel assigns to this series as its generalized sum that admits an expansion of the form $S(z)$.

Thus a special class of the Ω-series is in close relation with one of the most interesting questions of analysis, namely, the question as to what we shall mean by the sum of a divergent series.

Moreover the function defined by the series $S(z)$ is asymptotic to the power series of which it is the Borel sum; and another interesting point of connection for series $\Omega(x)$ thus emerges.

In later memoirs I intend to exhibit these and other properties of the series $S(z)$ in their place as special cases of properties possessed by the Ω-series in general.

14. Again, if we put Mz for x and $\alpha + 1$ for b/a in the Ω-

series of the preceding section, we obtain a new series which is readily reduced to the form

$$(7)\quad \overline{S}(z) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{(Mz+\alpha+1)(Mz+\alpha+2)\cdots(Mz+\alpha+n)}.$$

This series plays the leading rôle in Watson's fundamental paper referred to in section 2. Watson shows that most of the ordinary functions of analysis (which possess asymptotic expansions) are capable of convergent representations in the form of series $\overline{S}(z)$. This statement is justified by three leading results which are stated by him as Theorems I, II, III. Through such results one sees from another point of view the importance of our series $\Omega(x)$.

15. In a suggestive paper on the solutions of homogeneous linear differential equations in the neighborhood of the irregular point infinity Horn (loc. cit.) has made use of a series which he writes in the form

$$(8)\quad A_0 + \sum_{n=1}^{\infty} (A_n + B_n x) \times \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}{(x^2/k+\lambda+1)(x^2/k+\lambda+2)\cdots(x^2/k+\lambda+n)}.$$

He treats the equation

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0,$$

in which the coefficients P_0, P_1, P_2 are polynomials in x of degrees m, $m+1$, $m+2$ such that the characteristic algebraic equation for the point infinity of the differential equation has its roots distinct. For such an equation the point infinity is irregular of rank 2. Horn shows that the solutions of such an equation can be represented by means of convergent series (8) and he points out that the results are capable of generalization to equations of any order and with any rank of irregular point at infinity.

It is thus apparent that series (8) are of great importance in the theory of differential equations.

We shall now exhibit series (8) as directly dependent on a special case of our series $T(x)$. For this purpose we choose for $g(x)$ the value

$$g(x) = \frac{1}{\Gamma(x+\lambda+1)},$$

so that we have

$$\frac{g(x+n)}{g(x)} = \frac{1}{(x+\lambda+1)(x+\lambda+2)\cdots(x+\lambda+n)}.$$

Thence we may write

$$\Omega_1(x^2) = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{(x^2+\lambda+1)\cdots(x^2+\lambda+n)},$$

$$\Omega_2(x^2) = \sum_{n=1}^{\infty} \frac{d_n}{(x^2+\lambda+1)\cdots(x^2+\lambda+n)}.$$

Multiplying the first of these series by 1 and the second by x and adding, we have

$$xT(x) = c_0 + \sum_{n=1}^{\infty} \frac{c_n + d_n x}{(x^2+\lambda+1)\cdots(x^2+\lambda+n)},$$

where $T(x)$ denotes an instance of the series $T_{\nu_i}(x)$ defined in section 6. If we now replace x by $x/\sqrt{k}$ it is clear that we have a series of the form (8).

Thus we have exhibited several series of analysis as special cases of our general class of Ω-series; in this way we have tentatively shown the importance of the latter series.

Some Other Special Cases of Interest.

16. Let us consider the first order difference equation

$$g(x+1) = R(x)g(x) \equiv a\frac{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)}{(x-\beta_1)(x-\beta_2)\cdots(x-\beta_k)}g(x),$$

in which the order $m-k$ of $R(x)$ at infinity is different from zero. This equation obviously has a solution of the form

$$g(x) = a^x\frac{\Gamma(x-\alpha_1)\cdots\Gamma(x-\alpha_m)}{\Gamma(x-\beta_1)\cdots\Gamma(x-\beta_k)}.$$

From the asymptotic representation of $\Gamma(x)$, quoted above, it follows readily that the foregoing function $g(x)$ has an asymptotic representation of the form

$$g(x) \sim x^{\mu x+\nu} e^{\alpha x+\beta}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right), \quad \mu = m-k.$$

Hence the function $g(x)$ is suitable for use in defining Ω-series.

It is clear that we have

$$\frac{g(x+n)}{g(x)} = \frac{g(x+1)}{g(x)} \cdot \frac{g(x+2)}{g(x+1)} \cdot \cdots \cdot \frac{g(x+n)}{g(x+n-1)}$$
$$= R(x)R(x+1)\cdots R(x+n-1).$$

Hence our series $\Omega(x)$ in this case takes the form

$$\Omega(x) = c_0 + \sum_{n=1}^{\infty} c_n R(x)R(x+1)\cdots R(x+n-1).$$

This form of series $\Omega(x)$ essentially includes as special cases the three types of series mentioned in sections 12–14. It affords a highly interesting class of expansions in rational functions, particularly when $R(x)$ vanishes to the first order at infinity, as I intend to show in a future paper.

Let us consider the special case in which $R(x)$ vanishes at infinity to the integral order $s+1$. Then $\mu = -s-1$. For this case we have a particularly useful class of series $T(x)$, namely,

$$T(x) = c_{00} + \frac{c_{01}}{x} + \cdots + \frac{c_{0s}}{x^s}$$
$$+ \sum_{n=1}^{\infty}\left(c_{n0} + \frac{c_{n1}}{x} + \cdots + \frac{c_{ns}}{x^s}\right) R(x)R(x+1)\cdots R(x+n-1).$$

It is not difficult to show that this series may be formally transformed into a descending power series

$$\beta_0 + \frac{\beta_1}{x} + \frac{\beta_2}{x^2} + \cdots;$$

and conversely that any such power series may be formally transformed into a series $T(x)$ depending on a given $R(x)$ vanishing at infinity to the order $s+1$. The transformation is unique in each direction. Moreover the power series may diverge for all values of x while the T-series has a half-plane of convergence. It is in connection with such facts that these T-series exhibit their great importance, as I shall show in detail in a later paper.

17. A certain subclass of the functions $g(x)$ employed in the preceding section is of special interest on account of their applications. This subclass consists of those functions $g(x)$ in which $g(x)/g(x+1)$ is a rational function of x and $g(x)/g(x+n)$ is a polynomial in x provided that n is sufficiently large. A useful more restricted class is that in which this polynomial has non-negative real coefficients. We shall now exhibit a few such functions $g(x)$.

We take first a case already treated, namely, that in which

$$g(x) = \frac{1}{a^x\Gamma(x+b/a)}, \quad a \neq 0.$$

Here $R(x)$ has the value $1/(ax+b)$. Hence $g(x)/g(x+n)$ is a polynomial in x of degree n. Moreover, its coefficients are non-negative and real if a is positive and b is non-negative and real.

Again, we may take

$$g(x) = \frac{\Gamma(x+1)}{\Gamma(x)\Gamma(x+3)}.$$

Here it is easy to see that $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large, in fact, provided that n is greater than 2.

In a similar way one may treat the function

$$g(x) = \frac{\Gamma(x+1)\Gamma(x+3)}{\Gamma(x)\Gamma(x+2)\Gamma(x+4)}$$

and with a like result.

In the foregoing cases $R(x)$ vanishes to the first order at infinity. In the following it vanishes to the second order:

$$g(x) = \frac{\Gamma(x+1)}{\Gamma(x)\Gamma(x+3)\Gamma(x+4)},$$

$$g(x) = \frac{\Gamma(x+1)\Gamma(x+3)}{\Gamma(x)\Gamma(x+2)\Gamma(x+4)\Gamma(x+5)}.$$

It is obvious in what way one may form other functions $g(x)$ so that $R(x)$ vanishes at infinity to any desired order while $g(x)/g(x+n)$ has the desired polynomial character.

18. If we take for $g(x)$ a function of the general type of those defined by difference equations in accordance with the

theorem quoted in section 7, namely,

$$(9) \qquad g(x) \sim x^{-(s+1)x+\mu} e^{\alpha x}\left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right),$$

it is easy to see that $g(x+n)/g(x)$ has an asymptotic representation of the form

$$\frac{g(x+n)}{g(x)} \sim x^{-(s+1)n}\left(\beta_{0n} + \frac{\beta_{1n}}{x} + \frac{\beta_{2n}}{x^2} + \cdots\right).$$

If we take for the value of s a non-negative integer this function $g(x)$ gives rise to the T-series

$$(10) \qquad T(x) = P_{0\nu}(x) + \sum_{n=1}^{\infty} P_{n\nu}(x) \frac{g(x^t+n)}{g(x^t)},$$

where

$$P_{n\nu}(x) = c_{n0} + \frac{c_{n1}}{x} + \frac{c_{n2}}{x^2} + \cdots + \frac{c_{n\nu}}{x^\nu} \qquad (n = 0, 1, 2, \cdots).$$

If we take $\nu = st + t - 1$ it is easy to see that this T-series may be formally transformed uniquely into a descending power series

$$\beta_0 + \frac{\beta_1}{x} + \frac{\beta_2}{x^2} + \cdots,$$

and conversely that any such descending power series may be formally transformed uniquely into a series $T(x)$ defined by means of any basic function $g(x)$ having the asymptotic character (9).

A special case of the foregoing series of particular interest is that depending on a function $g(x)$ for which in (9) we have $\mu = \alpha = 0 = c_1 = c_2 = c_3 = \cdots$.

It may be added that the function $T(x)$ defined by (10) is asymptotic to the power series into which the series $T(x)$ is transformable; a proof will be given in a later paper.

Character of the Regions of Convergence.

19. In my *Transactions* paper referred to in § 3 above I have proved for the general series $\Omega(x)$ in (3) the following theorem:

If σ denotes α_m or μ_h according as m is or is not greater than

k, then there exists a unique real number λ $[\mu]$ such that the region of convergence (absolute convergence) (exclusive of the exceptional points) of the series $\Omega(x)$ is bounded by the straight line $R(\sigma x) = \lambda$ $[R(\sigma x) = \mu]$ and lies on that side of this line for which $R(\sigma x) < \lambda$ $[R(\sigma x) < \mu]$.

We shall now extend this theorem to the case of the series $T(x)$.

In view of the theorem just quoted it is easy to see that the series

$$\Omega_i(x^t) = \sum_{n=0}^{\infty} c_{ni} \frac{g(x^t + n)}{g(x^t)} \tag{11}$$

has a region of convergence (absolute convergence) bounded by the curve

$$R(\sigma x^t) = \lambda_i \quad [R(\sigma x^t) = \mu_i],$$

where λ_i $[\mu_i]$ is an appropriately determined constant, and that it lies on that side of this curve for which

$$R(\sigma x^t) < \lambda_i \quad [R(\sigma x^t) < \mu_i].$$

Let us now consider the $\nu + 1$ series (11) for which i takes the values $0, 1, 2, \cdots, \nu$. Multiplying $\Omega_i(x^t)$ by $1/x^i$ $(i = 0, 1, \cdots, \nu)$ and adding the resulting series term by term we have

$$T_{\nu t}(x) = \sum_{n=0}^{\infty} P_{n\nu}(x) \frac{g(x^t + n)}{g(x^t)}, \tag{12}$$

where

$$P_{n\nu}(x) = c_{n0} + \frac{c_{n1}}{x} + \cdots + \frac{c_{n\nu}}{x^\nu}.$$

Let ρ $[\tau]$ be a number of the set $0, 1, 2, \cdots, \nu$ such that

$$\lambda_\rho \leqq \lambda_i,\ i = 0, 1, \cdots, \nu,\ i \neq \rho \quad [\mu_\tau \leqq \mu_i,\ i = 0, 1, \cdots, \nu,\ i \neq \tau].$$

Then it is clear that $T_{\nu t}(x)$ converges (converges absolutely) for any value of x such that

$$R(\sigma x^t) < \lambda_\rho \quad [R(\sigma x^t) < \mu_\tau]. \tag{13}$$

If furthermore $\lambda_\rho < \lambda_i$ when $i \neq \rho$, then the region of convergence of $T_{\nu t}(x)$ is bounded by the curve

$$R(\sigma x^t) = \lambda_\rho.$$

If on the other hand λ_ρ $[\mu_\tau]$ is equal to some other λ $[\mu]$, then

$T_{\nu t}(x)$ may converge (converge absolutely) for certain non-exceptional values of x outside of the region defined by (13). This is put in evidence by the series

$$\sum_{n=0}^{\infty}\left[\left(\frac{1}{n!}+2\right)+x\right]\frac{\Gamma(x)\Gamma(n)}{\Gamma(x+n)}.$$

Here $\lambda_0 = \mu_0 = \lambda_1 = \mu_1 = 1$, as one may show without great difficulty by making use of the asymptotic character of $\Gamma(x)$. This series converges absolutely for the non-exceptional value $x = -2$ although both relations in (13) now reduce to $R(-x) < 1$ or $R(x) > -1$.

In view of this possibility that $T_{\nu t}(x)$ may converge (converge absolutely) outside of the common part of the regions of convergence (absolute convergence) of the component series (11) it seems desirable to state the following theorem, the proof of which is immediate in view of relation (10) in my *Transactions* paper already referred to:

Let x_0 and x_1 be two values of x which are non-exceptional for the series $T_{\nu t}(x)$ and suppose that $T_{\nu t}(x_0)$ converges absolutely. Then $T_{\nu t}(x_1)$ converges absolutely if $R(\sigma x_1{}^t) < R(\sigma x_0{}^t)$ and the quotient

$$P_{n\nu}(x_1)/P_{n\nu}(x_0)$$

is bounded.

20. We should examine briefly the nature of the curves

$$R(\sigma x^t) = \eta \tag{14}$$

which bound the regions of convergence and absolute convergence of series $T_{\nu t}(x)$. For this purpose we write

$$\sigma = |\sigma| e^{\phi\sqrt{-1}}, \quad x = re^{\theta\sqrt{-1}},$$

where r is real and not negative. Then (14) reduces readily to the relation

$$r^t \cos(t\theta + \varphi) = \eta_1 \tag{15}$$

where

$$\eta_1 = \eta/|\sigma|.$$

In case $\eta_1 = 0$ equation (15) represents $2t$ rays from zero to infinity and dividing the space about zero into $2t$ equal parts or sectors. The quantity denoted by the first member of (15) is negative within alternate sectors of this set (the sectors of convergence) and positive within the others (in

general the sectors of divergence). If η_1 is negative then (15) represents a curve of t branches lying within the aforementioned sectors within which the first member of (15) denotes a negative quantity; and each branch approaches asymptotically the rays of the sector including it. In this case the region of convergence consists in general of t separated regions each bounded by a curve somewhat resembling one branch of an hyperbola. In case η_1 is positive the branches of the curve denoted by (15) lie in the aforementioned sectors within which the first member of (15) is positive; and again each branch approaches asymptotically the rays of the sector including it. In this case the region of convergence is in general the portion of the plane excluded by the t branches, which again resemble branches of hyperbolas.

UNIVERSITY OF ILLINOIS,
April 14, 1917.

SHORTER NOTICE.

Interpolated Six-place Tables. Edited by HORACE WILMER MARSH. New York, John Wiley and Sons, 1916. xii + 155 pp.

THIS volume contains logarithms of numbers and the natural and logarithmic trigonometric functions, also tables of length, area, volume, weight, metric conversion, decimal equivalents, and specific gravity. In view of the many logarithmic tables on the market, a new compilation is expected to present valuable improvements. Only one such is evident in this book, viz., the use of a heavy ruling to denote "the change in leading figures when occurring in the line, thereby making the use of the wrong leading figures possible only by 'jumping the fence.'" Unfortunately on page 39, an omitted "fence" gives wrong values for log tan 3° 0′, 20″, 30″, 40″, and 50″.

In the logarithms of numbers no horizontal spacings or rulings are used and the number of rows on a page varies from 13 to 31, so that the position of the desired logarithm on the page is never known "a priori." In the logarithmic trigonometric tables the lines are separated into groups of ten for 6 pages only, though this grouping is used throughout the

table of natural functions. In no case is the last line of a page reproduced on the next page, so that in numerous cases one portion of a mantissa is at the bottom of the page and the remainder on the next page, in one instance requiring the leaf to be turned.

The natural function tables read down the page, while the log function tables read across, making the differences for the latter difficult of computation mentally. In the preface on page v, exponent is used for power, and 10^5 is given as 10000, also on page vii log 83.19 should be 1.920071. On page ix is found the objectionable form of negative characteristic with positive mantissa, also the symbol of identity to express "whose anti-log equals." On page x the reader learns that a negative number has no logarithm and that there are two kinds of logarithms, tabular and non-tabular; finally in these tables log cos 72° 25′ must be taken as log cos 72° 24′ 60″.

These tables are characterized by the almost complete absence of mechanical aids to the eye and are entirely unsuited to the use of the student or professional computer.

The following corrections should be noticed: broken type in log tan 37° 48′ 50″; log 8.140 should be .910624, a correction in the second digit which affects 170 logarithms following; log 8.760 should be .942504, similarly affecting 110 logarithms following. Isolated corrections are:

log sin 24° 52′ 20″	9.623865	log tan 65° 3′ 20″	10.332428
" " 31° 22′ 10″	9.716466	" " 68° 34′ 20″	10.406210
" " 38° 38′ 50″	9.795549	nat sin 46° 1′	.719542
" " 48° 54′ 50″	9.877212	" " 59° 3′	.857616
" " 50° 2′ 10″	9.884483	" tan 57° 36′	1.57575
" " 61° 3′ 0″	9.942029	" " 80° 6′	5.72974.
" " 81° 14′ 30″	9.994906		

F. H. SAFFORD.

NOTES.

THE April number (volume 18, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Differential equations and implicit functions in infinitely many variables," by W. L. HART; "On the equivalence of écart and voisinage," by E. W. CHITTENDEN;

"On the theory of associative division algebras," by OLIVE C. HAZLETT; "The converse of the theorem concerning the division of a plane by an open curve," by J. R. KLINE; "On the conformal mapping of curvilinear angles. The functional equation $\phi[f(x)] = a_1\phi(x)$," by G. A. PFEIFFER; "Dynamical systems with two degrees of freedom," by G. D. BIRKHOFF.

THE April number (volume 39, number 2) of the *American Journal of Mathematics* contains: "Fourier's constants of functions of several variables," by W. W. KÜSTERMANN; "Equations involving the partial derivatives of a function of a surface," by C. A. FISCHER; "Invariants and covariants of the Cremona cubic surface," by C. P. SOUSLEY; "The lines of electric force due to a moving electron," by F. D. MURNAGHAN; "On inequalities of certain types in general linear integral equation theory," by MARY E. WELLS; "A trigonometrical sum and the Gibbs' phenomenon in Fourier's series," by H. S. CARSLAW; "On the relation between some important notions of projective and metrical differential geometry," by F. M. MORRISON.

THE following advanced courses in mathematics are announced for the year 1917–1918.

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Differential equations, four hours.—By Professor F. N. COLE: Theory of groups, three hours; Invariants and higher plane curves, three hours, first half-year.—By Professor JAMES MACLAY: Theory of geometric constructions, three hours, first half-year; Elliptic functions, three hours, first half-year.—By Professor C. J. KEYSER: Modern theories in geometry, four hours; Mathematics, three hours, second half-year.—By Professor D. E. SMITH: History of mathematics, two or three hours.—By Professor EDWARD KASNER, Seminar in differential geometry, two hours; (with Dr. C. A. FISCHER) Theory of functionals and integral equations, three hours, first half-year.—By Professor W. B. FITE: Differential equations, three hours, second half-year.—By Professor H. E. HAWKES: Differential geometry of curves, three hours, second half-year.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of probabilities, three hours; Introduction to actuarial science, three hours.—By Professor V. SNYDER: Projective geometry,

three hours.—By Professor F. R. SHARPE: Vector analysis with applications to physics, three hours (first term).—By Professor W. B. CARVER: Elementary theory of groups, three hours (second term); Synopsis of higher mathematics, three hours (with Dr. SILVERMAN).—By Professor A. RANUM: Differential geometry, three hours (first term).—By Professor D. C. GILLESPIE: Advanced calculus, three hours.—By Professor W. A. HURWITZ: Differential equations of physics, three hours.—By Professor C. F. CRAIG: Fourier series and the potential function, three hours; Teachers' course in mathematics, three hours.—By Professor F. W. OWENS: Mathematical physics, three hours.—By Dr. L. L. SILVERMAN: Infinite series, three hours.—By Dr. J. V. MCKELVEY: Algebraic curves, three hours.—By Mr. H. BETZ: Elementary differential equations, three hours.—By Dr. M. G. GABA: Theory of equations, three hours (first term).—By Dr. R. E. GILMAN: Advanced analytic geometry, three hours.

HARVARD UNIVERSITY.—By Professor W. F. OSGOOD: Advanced calculus, part II (second term), three hours; Infinite series and products (first term), three hours; Theory of functions, second course, three hours.—By Professor M. BÔCHER: Introduction to modern geometry and modern algebra, three hours; Algebra (second term), three hours.—By Professor C. L. BOUTON: Elementary differential equations (second term), three hours; Differential equations and Lie's theory, three hours.—By Professor J. L. COOLIDGE: Subject matter of elementary mathematics (first term), three hours; Probability (second term), three hours; Algebraic plane curves, three hours.—By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics (first term), two or three hours.—By Professor H. N. DAVIS: Dynamics, three hours.—By Professor G. D. BIRKHOFF: Vector analysis (first term), three hours; Theory of heat and elastic vibrations (second term), three hours; Integral equations (first term), three hours.—By Professor D. JACKSON: Advanced calculus, part I (first term), three hours; Introduction to potential functions and Laplace's equation (first term), three hours; Developments in series (second term), three hours.—By Dr. G. M. GREEN: Theory of functions, three hours.—By Drs. G. M. GREEN and W. LEROY HART: Differential geometry, three hours.—By Dr. W. LEROY HART: Introduction to celestial mechanics (second term),

three hours.—By Dr. T. A. PIERCE: Theory of numbers (first term), three hours; Algebraic numbers (second term), three hours.

Professor Birkhoff will conduct a fortnightly seminary in analysis.

Courses of research are also offered by Professor Osgood in the theory of functions, by Professor Bôcher in the real solutions of linear differential equations, by Professor Bouton in the theory of point transformations, by Professor Coolidge in geometry, by Professor Birkhoff in the theory of differential equations, by Professor Jackson in the theory of functions of a real variable and by Dr. Green in differential geometry.

UNIVERSITY OF ILLINOIS.—All courses are three hours for the year except as otherwise indicated.—By Professor E. J. TOWNSEND: Functions of a complex variable; Differential equations and advanced calculus.—By Professor G. A. MILLER: Elementary theory of groups; Theory of equations and determinants (first semester).—By Professor H. L. RIETZ: Theory of statistics.—By Professor J. B. SHAW: General algebra.—By Professor C. H. SISAM: Algebraic surfaces; Solid analytic geometry (second semester).—By Professor A. EMCH: Projective geometry; Constructive geometry (second semester).—By Professor R. D. CARMICHAEL: Theory of linear difference equations.—By Professor A. R. CRATHORNE: Theory of mathematical instruments (second semester).—By Dr. E. B. LYTLE: Teacher's course (two hours, first semester); History of mathematics (two hours, second semester).—By Dr. A. J. KEMPNER: Modern algebra.

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry, two hours; Theory of functions, two hours. —By Professor A. B. COBLE: Modular functions, two hours. —By Professor A. COHEN: Differential geometry, two hours; Theory of real functions, two hours.—By Dr. H. BATEMAN: Differential equations of physics, two hours.

Analytic mechanics, three hours.—By Dr. J. W. ALEXANDER: Algebraic functions, three hours.—By Dr. G. A. PFEIFFER: Theory of functions of real variables, three hours.

YALE UNIVERSITY.—By Professor E. W. BROWN: Advanced calculus, three hours; Advanced dynamics, two hours.—By Professor J. PIERPONT: Theory of functions of a complex variable, two hours; Elliptic functions, two hours.—By Professor P. F. SMITH: Differential equations, two hours.—By Professor W. R. LONGLEY: Integral equations, two hours (second term); Potential theory and harmonic analysis (first term).—By Professor E. J. MILES: Calculus of variations, two hours.—By Professor J. I. TRACY: Modern analytic geometry, two hours. —By Dr. D. F. BARROW: Advanced algebra, two hours.—By Mr. W. L. CRUM: Statics and dynamics, two hours.—By Mr. J. K. WHITTEMORE: Differential geometry, two hours.

DEAN F. C. FERRY, of Williams College, has been elected president of Hamilton College.

PROFESSORS R. C. ARCHIBALD, FRANK MORLEY, and T. LEVI-CIVITA have been elected fellows of the American academy of arts and sciences.

AT the Massachusetts Institute of Technology, Professor E. B. WILSON has been appointed head of the department of physics.

AT Brown University assistant professor R. C. ARCHIBALD has been promoted to an associate professorship of mathematics.

MR. C. C. CAMP has been appointed professor of pure mathematics at Ottawa University.

DR. L. T. WILSON, of the University of Illinois, Mr. W. E. [redacted] University and Dr. J. H. WEAVER, of

in mathematics at the University of Illinois. Mr. R. E. BILLINGS has been appointed graduate assistant in mathematics.

AT the University of Oklahoma, Dr. NATHAN ALTSHILLER has been promoted to an assistant professorship of mathematics.

DR. J. W. NICHOLSON, for forty years professor of mathematics in the University of Louisiana and author of numerous textbooks, died on March 22.

MR. W. C. WRIGHT, consulting actuary, died at his home in Medford, Mass., on April 23. Mr. Wright had been a member of the American Mathematical Society since 1898.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALONSO-MISOL (F.). Elementos de las teorías de funciones y derivadas. Libro I. 2a edición. Madrid, Fortanet, 1916. Pes. 15.00

BOREL (E.) and VOLTERRA (V.). Lectures delivered at the formal opening of the Rice Institute: Aggregates of zero measure and Monogenic uniform non-analytic functions by E. Borel and The generalization of analytic functions and On the theory of waves and Green's functions by V. Volterra. Translated into English by G. C. Evans and P. J. Daniell. (*Rice Institute Pamphlet*, Vol. 4, No. 1.) Houston, Rice Institute, 1917. 117 pp.

CAUCHY (A.). Œuvres complètes. 2e série, Tome 12: Nouveaux exercices d'analyse et de physique, Tome 2. Paris, Gauthier-Villars, 1916. 4to. 472 pp. Fr. 25.00

DANIELL (P. J.). See BOREL (E.).

EVANS (G. C.). See BOREL (E.).

GALDEANO (Z. G. DE). Tratado general de matemáticas. Zaragoza, Casañal, 1916. 8vo. 112 pp. Pes. 2.50

LEVI (B.). Introduzione alla analisi matematica. I: Teorie formali. Paris, Hermann; Parma, presso l'autore (Napoli, B. de Rubertis), 1916. 8vo. 482 pp. L. 15.50

LIBBY (W.). An introduction to the history of science. Boston, Houghton-Mifflin, 1917. 8vo. 12+288 pp.

LICKS (H. E.). Recreations in mathematics. New York, Van Nostrand, 1917. 8vo. 162 pp. $1.25

NELKENBRECHER (R.). Die Regelflächen, die durch Biegung aus den Hauptnormalflachen der Kurven konstanter Krümmung hervorgehen. (Diss., Halle-Wittenberg.) Halle, 1916. 72 pp.

REY PASTOR (J.). Resumen de las lecciones de análisis matemático (segundo curso). Curso de 1915–1916. Madrid, 1916. 4to. 8+472 pp. Pes. 12.00

STOILOW (S.). Sur une classe de fonctions de deux variables définies par les équations linéaires aux dérivées partielles. (Thèse.) Paris, Gauthier-Villars, 1916. 4to. 6+84 pp. Fr. 4.00

TEIXEIRA (F. G.). Obras sobre mathematica. Vol. 7. Coimbra, Imprensa da Universidade, 1915. 4to. 8+424 pp.

VEGAS (M.). Problemas de geometría analítica. Madrid, Fortanet, 1916.

VOLTERRA (V.). See BOREL (E.).

II. ELEMENTARY MATHEMATICS.

BORCHARDT (W. G.). Revision papers in arithmetic. London, Rivingtons, 1917. With answers. 2s.

COMBEROUSSE (C. DE). Arithmétique. (Tome 1 du Cours de mathématique.) 6e édition. Paris, Gauthier-Villars, 1916. 8vo. Fr. 4.00

FESTA (N.). See VACCA (G.).

MINGOT Y SHELLY (J.). Elementos de álgebra y trigonometría. Valladolid, A. Martin, 1915. 8vo.

PINCHERLE (S.). Algebra elementare. 12a edizione, riveduta. (Manuali Hoepli.) Milano, Hoepli, 1916. 16mo. 8+210 pp. L. 1.50

ROCHE (S. L.). Addition and subtraction, the domino method. Baltimore, Meyer and Thalheimer, 1916. 63 pp. $1.00

SAXELBY (C. H.). See SAXELBY (F. M.).

SAXELBY (F. M. and C. H.). Practical arithmetic and mensuration. London, Longmans, 1917. 2s. 6d.

SERRET (J. A.). Traité de trigonométrie. 10e édition. Paris, Gauthier-Villars, 1916. 8vo. 10+336 pp. Fr. 4.00

TAVOLE logarithmiche a cinque cifre decimali, raccolte e pubblicate per cura dell'Istituto idrografico della r. marina. Edizione ridotta. Genova, tip. Istituto idrografico della r. marina, 1916. 15+93 pp. L. 1.00

TAVOLE logarithmiche e nautiche, raccolte e pubblicate per cura dell'Istituto idrografica della r. marina. 2a edizione, riveduta, corretta ed ampliata. Genova, tip. Istituto idrografico della r. marina, 1916. 4to. 15+93+23+117 pp.

VACCA (G.). Euclide: il primo libro degli elementi. Testo greco, versione italiana. Con prefazione di Nicola Festa. Firenze, Sausoni, 1916. 8vo.

III. APPLIED MATHEMATICS.

ANGOT (A.). Traité élémentaire de météorologie. 3e édition. Paris, Gauthier-Villars, 1916. 8vo. 7+416 pp. Fr 14.00

AYDELOTTE (F.). English and engineering. A volume of essays for English classes in engineering schools. New York, McGraw-Hill, 1917. 8vo. 390 pp. $1.50

BANACHIEWICZ (T.). Tables auxiliaires pour la résolution de l'équation de Gauss $\sin(z-q) = m \sin^4 z$ dans la détermination d'une orbite planétaire, le nombre de décimales y étant conforme à l'exactitude du calcul de l'orbite à 6 décimales. Paris, Gauthier-Villars, 1916. 8vo. 20 pp. Fr. 2.50

——. Tables fondamentales pour la résolution de l'équation de Gauss $\sin(z-q) = m^4 \sin z$ dans la détermination d'une orbite planétaire, le nombre de décimales y étant conforme à l'exactitude du calcul de l'orbite à 7 décimales. Paris, Gauthier-Villars, 1916. 4to. 28 pp. Fr. 5.00

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and ninety-first regular meeting of the Society was held in New York City on Saturday, April 28, 1917, extending through the usual morning and afternoon sessions. The attendance included the following twenty-seven members:

Mr. D. R. Belcher, Professor E. G. Bill, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. A. Garrison, Professor O. E. Glenn, Dr. Olive C. Hazlett, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor P. H. Linehan, Professor W. R. Longley, Professor R. L. Moore, Mr. G. W. Mullins, Dr. Alexander Pell, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor Oswald Veblen, Dr. Mary E. Wells, Mr. J. K. Whittemore.

Ex-President E. W. Brown occupied the chair at the morning session, being relieved at the afternoon session by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Professor C. F. F. Garis, Union College; Professor F. J. Holder, University of Pittsburgh; Dr. V. H. Wells, University of Michigan; Professor W. L. Wright, Lincoln University, Pa. Six applications for membership in the Society were received.

Professor L. P. Eisenhart was reelected a member of the Editorial Committee of the *Transactions*, to serve until October 1, 1920. Professor E. R. Hedrick was appointed delegate of the Society to attend the inauguration of President Jessup of the State University of Iowa on May 11–12. A committee consisting of Professors Fite, Birkhoff, and Veblen was appointed to consider and report to the Council any measures which it may be desirable to take to increase the interest and efficiency of the New York meetings of the Society.

Committees were also appointed to consider the question of the legal incorporation of the Society and to prepare a list of nominations of officers and other members of the Council to be elected at the annual meeting in December.

The following papers were read at this meeting:

(1) Professor W. B. FITE: "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivatives."

(2) Dr. SAMUEL BEATTY: "The inversion of an analytic function."

(3) Professor MAURICE FRÉCHET: "Relations entre les notions de limite et de distance."

(4) Professor O. E. GLENN: "A fundamental system of formal covariants mod 2 of the binary cubic."

(5) Professor LUIGI BIANCHI: "Concerning singular transformations B_k of surfaces applicable to quadrics."

(6) Professor J. E. ROWE: "The projection of a line section upon the rational plane cubic curve."

(7) Mr. L. B. ROBINSON: "On partial differential equations which define certain covariants."

(8) Mr. J. K. WHITTEMORE: "Kinematic properties of ruled surfaces."

(9) Dr. OLIVE C. HAZLETT: "On Huntington's set of postulates for abstract geometry."

(10) Mr. E. F. SIMONDS: "Differential invariants in the plane."

(11) Mr. JESSE DOUGLAS: "On certain two-point properties of doubly infinite families of curves on an arbitrary surface."

(12) Professor L. P. EISENHART: "Conjugate planar nets with equal invariants."

(13) Dr. ALEXANDER PELL: "Solutions of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and their application."

(14) Dr. C. A. FISCHER: "On bilinear and n-linear functionals."

(15) Professor E. B. WILSON: "Classification of real strains in hyperspace."

(16) Professor F. H. SAFFORD: "Irrational transformations of the general elliptic element."

(17) Dr. J. H. WEAVER: "Some algebraic curves."

(18) Professor R. L. MOORE: "A necessary and sufficient condition that a sequence of simple arcs of specified type should be equivalent, from the standpoint of analysis situs, to a sequence of straight segments."

(19) Professor DUNHAM JACKSON: "Second note on the parametric representation of an arbitrary continuous curve."

(20) Professor DUNHAM JACKSON: "Roots and singular points of semi-analytic functions."

(21) Professor OSWALD VEBLEN: "Doubly oriented lines."

(22) Dr. G. M. GREEN: "The intersections of a straight line and a hyperquadric."

(23) Dr. F. W. BEAL: "On a congruence of circles."

(24) Professor G. A. MILLER: "Possible characteristic operators of a group."

(25) Professor R. D. CARMICHAEL: "Examples of a remarkable class of series."

(26) Dr. W. L. HART: "Note on infinite systems of linear equations."

Professor Fréchet's paper was communicated to the Society by Professor D. R. Curtiss, Professor Bianchi's through Professor Eisenhart. Mr. Simonds and Mr. Douglas were introduced by Professor Kasner. Professor Bianchi's paper was read by Professor Eisenhart, and the papers of Dr. Beatty, Professor Fréchet, Professor Rowe, Mr. Robinson, Professor Wilson, Professor Safford, Dr. Weaver, Dr. Green, Dr. Beal, Professor Miller, Professor Carmichael, and Dr. Hart were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Fite's paper deals with differential equations of the second and third orders and with certain ones of general order. The results are analogous to those given in a paper under the same title in the June number of the *Annals of Mathematics*. There may be cited in particular a theorem setting forth conditions under which a solution of an equation of the third order cannot vanish more than twice between two successive roots of another solution.

2. Dr. Beatty's paper appeared in full in the May BULLETIN.

3. Professor Fréchet's paper deals with classes (L) of abstract elements for which the limit of a sequence is defined, and determines conditions to be added such that the class (L) will be a class (D) admitting a definition of distance, but preserving the convergence ideas already adopted. The paper will appear in the *Transactions*.

4. Professor Glenn gives, in this paper, a complete system of covariants mod 2 of a binary cubic form having arbitrary

coefficients. The system is finite. The methods of generation and proof of the completeness of the set of fundamental covariants (which includes also five pure invariants) are developed from the point of view emphasized in the author's paper in volume 17 of the *Transactions* (page 545).

5. A plane and a point on it constitute a *facette*, the point being called the *center*. The tangent planes to a surface and the respective points of contact afford ∞^2 facettes which may be said to constitute the surface. Suppose we have a surface S and with each of its facettes f we assume associated ∞^1 facettes f' in such a way that the center of each facette f' lies in the corresponding plane of f and its plane passes through the center of f. In this way we get ∞^3 facettes f' which ordinarily cannot be coordinated into ∞^1 surfaces S'. It will be all the more unusual if the latter circumstance is satisfied by the ∞^3 facettes f' obtained when S undergoes a deformation by flexure and the facettes f' are carried along in invariable relation with respect to the facettes f. Suppose that for S_0, one deform of S, all the centers of the facettes f' associated with each f lie on the line of intersection of the plane of f and a fixed plane π. Professor Bianchi has proposed and solved the problem: To find what must be assumed concerning S_0 and its relation with the fixed plane π in order that for each deform of S_0 the ∞^3 facettes can be coordinated into ∞^1 surfaces S'. He shows that S_0 must be a quadric; that when it is not a quadric of revolution the fixed plane must be a principal plane and that the planes of the facettes f', associated with each facette f of S_0, envelope a cone projecting from the center of f, the focal conic of S_0 lying in the given diametral plane; when it is a quadric of revolution, the fixed plane must be a meridian plane, and the planes must form two pencils whose axes are the joins of the center of f with the foci of the meridian conic. These geometrical configurations had been obtained by Professor Bianchi formerly in his researches on the transformations of the deforms of quadrics. The paper will be published in the *Transactions*.

6. Professor Rowe's paper appeared in full in the June BULLETIN.

7. Given a differential system

$$A_{11}\frac{\partial f}{\partial x_2}+A_{12}\frac{\partial f}{\partial x_3}+\cdots+A_{1n}\frac{\partial f}{\partial x_{n+1}}=0,$$

$$A_{21}\frac{\partial f}{\partial x_2}+A_{22}\frac{\partial f}{\partial x_3}+\cdots+A_{2n}\frac{\partial f}{\partial x_{n+1}}=0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_{m1}\frac{\partial f}{\partial x_2}+A_{m2}\frac{\partial f}{\partial x_3}+\cdots+A_{mn}\frac{\partial f}{\partial x_{n+1}}=0,$$

where

A_{i1} are functions of x_1 only,

A_{i2} are functions of x_1, x_2 only,

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

A_{in} are functions of $x_1, x_2, \cdots, x_n$, but not of x_{n+1}

$$(i = 1, 2, \cdots, m);$$

Mr. Robinson shows that the singular points of such a system can always be determined in advance and that if the coefficients be polynomials the set can be integrated by a finite number of operations. The equations which define the covariants of systems of ordinary differential equations can always be reduced to such a system. The paper will be published in the next issue of the *Johns Hopkins Circular*.

8. In Mr. Whittemore's paper a study of some properties of ruled surfaces is made by reference of the surface to the moving triedral attached to any curve on the surface. Some properties of the general ruled surface are obtained, the most interesting of which is this: Any ruled surface may be generated by a radius fixed in a sphere whose center moves with unit velocity along the line of striction and which turns about the tangent to the line of striction with angular velocity equal to the reciprocal of the parameter of distribution of the surface; it follows that generators of two ruled surfaces, having the same line of striction and the same parameter of distribution, drawn through a point of the line of striction make a constant angle. Some particularly simple results are obtained for ruled surfaces whose parameters of distribution are equal to the radius of torsion of the line of striction.

9. In 1913, Professor Huntington* gave a set of postulates

* *Math. Annalen*, vol. 73, pp. 522–559.

for ordinary euclidean three-dimensional geometry, in terms of a class K of undefined elements, called "spheres," and the relation R of inclusion. At the time, he thought that the "general laws" were independent; but more recently, he noticed that Postulates 7 and 16 are redundant, and raised the question as to whether any others might be redundant.

Dr. Hazlett's paper shows that several other postulates are redundant, of which may be mentioned Postulates 4 and 6, which are special cases of Postulate 15. Furthermore, if Postulate 15 is satisfied, the vertices of a tetrahedron cannot be coplanar unless two or more coincide, and, in particular, the vertices of a triangle cannot be collinear unless two coincide. A slight change of wording, however, obviates this difficulty. Finally, the triangle transverse axiom (which Professor Huntington proved by the aid of the "general laws" 1–11 and the existence postulates E_1–E_3) is proved without the aid of any postulates beyond the first ten general laws.

10. Mr. Simonds' paper deals with the invariants under groups of point and contact transformations of differential configurations consisting of more than one regular analytic element. Previous papers bearing on this subject are P. Rabut: "Théorie des invariants universels," *Journal de l'Ecole Polytechnique*, 1898; E. Kasner: "Differential elements of the second order, etc.," *American Journal*, 1906, the latter pointing out an error in the former.

The following are the principal results of the present paper:

1. Under a finite continuous group of point transformations there are no *essential* invariants other than those of *two* elements.

2. (*a*) Under an infinite group of point transformations a single regular element has no invariants of order greater than zero. (*b*) There are, however, more general configurations having invariants. (*c*) In the case of certain imprimitive groups the simplest configurations having invariants present some exceptional features. (*d*) If λ_n be the smallest number of elements in the simplest configuration having invariants, $\lambda_n = 2n + 2$ for the entire group, $\lambda_n = n + 3$ for the area-preserving group, $\lambda_n = 3$ for the equilong group. The result for the entire group agrees with Kasner, and not with Rabut. Most of Rabut's results are wrong.

3. A set of theorems on contact transformations corresponding to 1 and 2 are obtained.

11. Mr. Douglas's paper belongs to what may be called "geometry in the neighborhood of a curve," as distinguished from the usual differential geometry which relates chiefly to the neighborhood of a point. Every question in this theory leads analytically to a problem in linear differential equations. This fact is a particular aspect of the principle of Darboux's "auxiliary equation" or Poincaré's "equations of variation," namely, that given an *arbitrary* system of differential equations, the solutions infinitely near to a given one are controlled by a system of *linear* differential equations.

In the present paper, a general doubly infinite curve family F on an arbitrary surface is studied with regard to certain properties involving two points a and b, a considerable distance apart. Let the curve C of the family F join a and b. Suppose at b an element of length $d\sigma$ placed normal to C, and its extremities joined to a by the curves C', C'' of F, these enclosing at a the infinitesimal angle $d\omega$. The ratio $d\sigma/d\omega$ is denoted by $V(a, b)$. When a and b are interchanged in the above construction the value obtained for the ratio $d\sigma/d\omega$ is denoted by $V(b, a)$. A general formula is established which expresses $V(a, b)/V(b, a)$ in terms of a certain integral taken along the curve C from a to b. One of the results flowing from this formula is that the doubly infinite curve families F for which everywhere $V(a, b) = V(b, a)$ are those for which the geodesic curvature is the same, at any given point, for all the curves of the family through that point. This is a generalization of a theorem of Levi-Civita to the effect that the geodesics themselves have the property $V(a, b) = V(b, a)$. A well-known theorem of Straubel on optical families is also generalized and found to hold for a wider category of curve families. The considerations of this paper have been extended to higher dimensions.

12. Koenigs has shown by geometrical considerations that the perspectives of asymptotic lines of a surface from a point on a plane form a conjugate net with equal invariants, and he observed, conversely, that such a planar net is always the projection of asymptotic lines on a surface. He stated that the converse problem is reducible to quadratures. Professor Eisenhart has shown that the methods of ordinary differential equations can be applied to this converse problem, and has found the coordinates of the surface in a form which is a generalization of the formulas of Lelieuvre.

13. In Dr. Pell's paper solutions of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ are obtained which lead to the representation of the coordinates of any curve as the sum of the coordinates of an arbitrary minimal curve and those of a curve lying on the minimal cone, whose arc is equal to the arc of the given curve. The necessary and sufficient conditions that the curve be algebraic are very simple. Some interesting results are easily obtained.

14. It has been proved by Fréchet that a bilinear functional $U[f(s), \varphi(t)]$ can be put in the form

$$\int\int f(s)\varphi(t)d_2u(s, t),$$

where $u(s, t)$ is regular in t, and after modifying the definition of the variation of a function of two variables he has proved that the variation of $u(s, t)$ is the least upper bound of the expression $|U[f, \varphi]|/mf\,m\varphi$, where mf and $m\varphi$ are the maxima of $|f(s)|$ and $|\varphi(t)|$. In the present paper Dr. Fischer has obtained the function $u(s, t)$ in a different way and proved it to be regular in both arguments and unique. Then the work can be extended by mathematical induction to n-linear functionals. In the last part of the paper it is proved that a homogeneous functional of the nth order can always be expressed as a multiple Stieltjes integral.

15. Professor Wilson shows that the algebraic method of Gibbs, which was used in an earlier communication (*Transactions of the Connecticut Academy of Sciences,* volume 14 (1907), pages 1–57) to classify dyadics or strains in hyperspace without regard to reality, affords a very easy means of carrying out the further classification with respect to reality. In addition to the tonics, the shears, and the tonic-shears, we find cyclotonics and cyclotonic-shears. The article will be printed in the *Journal of the Washington Academy of Sciences.*

16. Professor Safford continues in this paper a series of articles which have appeared in the BULLETIN and in the *Archiv der Mathematik und Physik.* These investigations are based upon a formula, published by G. G. A. Biermann and derived from Weierstrass's lectures, which expresses the solution of $[F'(x)]^2 = AF^4(x) + 4BF^3(x) + 6CF^2(x) + 4B'F(x) + A$ as an irrational function of $P(x)$.

17. If two lines l_1 and l_2 in the same plane pass through two fixed points, A and B, and rotate about A and B according to a definite law, the locus of the intersection of l_1 and l_2 will be a definite curve.

Dr. Weaver, by this means, has set up several curves and proved some of the relations existing between the fixed points and the curves.

In particular he has set up a curve C_n of degree n, which has at one of the fixed points an $(n-1)$ point, and such that the polar of the other fixed point with respect to C_n gives C_{n-1}.

18. Suppose that: (*a*) k and l are two parallel lines, (*b*) $A_1, A_2, A_3, \cdots$ and $C_1, C_2, C_3, \cdots$ are two sequences of points lying on k and l respectively and having, as sequential limit points, the points A and B respectively, (*c*) $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3, \cdots$, and ABC are simple continuous arcs no two of which have any point in common and all of which lie, except for their end points, entirely between k and l, (*d*) as $n \doteq \infty$ the arc $A_nB_nC_n$ approaches the arc ABC uniformly as its limit in the sense that if $\epsilon > 0$ there exists n_ϵ such that if $n > n_\epsilon$ then each point of $A_nB_nC_n$ is at a distance less than ϵ from some point of ABC.

Professor Moore proposes to show that a necessary and sufficient condition that such a set of arcs $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3, \cdots$ may be equivalent from the standpoint of analysis situs to a set of parallel segments of straight lines is that for every positive ϵ there should exist a positive δ_ϵ such that if n is any positive integer and X and Y are points of the arc $A_nB_nC_n$ and the distance from X to Y is less than δ_ϵ then that part of the arc $A_nB_nC_n$ which lies between X and Y lies entirely within some circle of radius less than ϵ.

19. In a paper recently presented to the Society (see BULLETIN, volume 23, page 68), Professor Jackson gave a proof of the theorem, apparently due originally to Fréchet, that an arbitrary continuous curve can be represented parametrically in such a form that the coordinates do not remain simultaneously constant throughout any interval of values of the independent variable. In the present note he points out that the theorem is practically obvious in the case of a rectifiable curve, and, by means of a generalization of the process which defines the length of such a curve, obtains a considerably simplified proof for the general case.

20. In his second paper, Professor Jackson shows how certain theorems about roots and singular points of analytic functions of two or more complex variables, Weierstrass's theorem of factorization, for example, can be extended, with appropriate modifications, to functions which are analytic in one of their arguments and merely continuous in all together. The treatment is based primarily on the use of contour integrals.

21. A doubly oriented line is a line in a three-space associated with one sense class among its points and one sense class among its planes. Professor Veblen's note deals with a number of theorems on order relations which cluster about this notion. It will be published in the second volume of Veblen and Young's Projective Geometry.

22. In Dr. Green's note, a simple proof is given of a formula for the intersections of a straight line and a hyperquadric which Professor Coolidge derived by an entirely different method in a paper communicated to the Society at the last annual meeting.

23. Dr. Beal's paper is concerned with a congruence C of circles which satisfies the following conditions. Any circle of the congruence lies in the tangent plane of a surface S and its center is the point of tangency M of the plane with S. At every point P of each circle a line is drawn which makes an angle φ with the plane of the circle. The projection of this line on the plane of the circle makes an angle ψ with the tangent to the circle at P. For any displacements of S in the neighborhood of M the point P is to move at right angles to this line. The surface on which P lies is a surface S_1 of a single parameter family of surfaces and is called a transform of S. The radius R of the circle and the angles φ and ψ are functions of u and v, the curvature coordinates of S. If the three quantities R, φ, and ψ are not restricted, congruences C exist for any surface S. The necessary and sufficient condition that lines of curvature correspond on the surface S and the surface S_1 is that R be constant. When ψ is a constant the principal radii of curvature t_1 and t_2 of any surface S_1 satisfy a very simple relation free from the angle θ which determines a particular surface S_1. When φ is a constant any surface parallel to S has associated with it a congruence C for which

the surfaces S_1 are parallel to the transforms S_1 of S. When R, φ, and ψ, are constants the surfaces S and S_1 are surfaces parallel to a pseudospherical surface and its Backlund transforms.

24. A characteristic operator of a group G is an operator which corresponds to itself in every possible automorphism of G. Every group contains at least one characteristic operator, viz., identity. A necessary and sufficient condition that an abelian group involves another characteristic operator is that it contains one and only one operator of order 2. In particular, an abelian group can never contain more than two characteristic operators. The main object of Professor Miller's paper is to prove the following theorem: It is possible to construct non-abelian groups whose characteristic operators constitute an arbitrary abelian group.

25. Professor Carmichael's paper appeared in full in the June BULLETIN.

26. Dr. Hart considers the infinite system of linear equations

$$\sum_{j=1}^{\infty} a_{ij}x_j = x_i' \qquad (i = 1, 2, \cdots), \tag{1}$$

where $\Sigma_{i=1}^{\infty}|x_i|^2$ and $\Sigma_{i=1}^{\infty}|x_i'|^2$ exist and where the infinite matrix $A = (a_{ij})_{i,j=1,2,\ldots}$ is limited. A condition is given under which (1) acts like a finite system of n equations in n unknowns; that is, a condition under which there exists an integer $w \geqq 0$ such that, for all $(x_1', x_2', \cdots)$ satisfying certain w linear relations, (1) possesses a solution depending linearly on w parameters. The condition is suggested by a property possessed by (1) when $A = E - B$ (E the unit matrix), where B is completely continuous in the Hilbert sense.

F. N. COLE,
Secretary.

THE MODULAR DIFFERENCE OF CLASSES.

BY PROFESSOR P. J. DANIELL.

In the general analysis of numerical functions the modular difference of two numbers is an extremely valuable concept. This paper discusses an analogous concept in the calculus of classes. These classes may be logical classes in general or may be point sets. The following notation* is used in which capital letters denote classes.

AB = logical product of A and B.

$A + B$ = logical sum of A and B.

$\bar{A}$ = complement of A or CA in Borel's notation.

$A < B$ means A is included in B.

$A > B$ means A includes B.

$A - B$ is defined only when $A > B$ and is then equal to $A\bar{B}$.

The modular difference of two numbers a, b satisfies the two conditions

(1) when $a \geqq b$, $| a - b | = a - b$,

(2) $| a - b |$ is symmetrical in a and b.

By analogy the modular difference of two classes A, B denoted by $| A - B |$ should satisfy

(1) when $A > B$, $| A - B | = A\bar{B} = A - B$,

(2) $| A - B |$ is symmetrical in A and B.

Definition.—The modular difference of two classes is the logical sum of the logical product of the first into the complement of the second and the logical product of the second into the complement of the first.

$$| A - B | = \bar{A}B + A\bar{B} \qquad \text{(Def.)}.$$

The reader is to note that the modular difference is defined even in cases where the difference according to Borel is not defined. That is to say, the modular difference is a direct operation on the two classes, not an operation on their difference.

Using the definition above and the analysis of symbolic logic the following properties can be proved.

* Borel, Théorie des Fonctions des Variables réelles, p. 16. Baldwin, Dictionary of Philosophy and Psychology, article "Symbolic Logic." Whitehead and Russell, Principia Mathematica, chapter on "Calculus of Classes."

Properties.

1(1). $|A-B| = A\overline{B}+\overline{A}B = (A+B)(\overline{A}+\overline{B}) = |\overline{A}-\overline{B}|$.

1(2). $|\overline{A-B}| = AB+\overline{A}\overline{B} = |\overline{A}-B| = |A-\overline{B}|$.

1(3). If $|A-B| = 0$, $A = B$ and vice versa.

(In this, 0 may be regarded as a set or class with no elements.)

2(1). $||A-B|-C| = |A-|B-C||$.

2(1) cor. If $|A-B| = C$, $A = |B-C|$.

2(2). $||A-B|-|B-C|| = |A-C|$.

2(2) cor. If $|A-B| = |B-C|$, $A = C$ and vice versa.

2(3). $|A-B||B-C||C-A| = 0$.

2(4). $C|A-B| = |CA-CB|$.

3(1). $|(A+B)-C| = |A-C||B-C|+|A-B|\overline{C}$.

3(2). $|AB-C| = |A-C||B-C|+|A-B|C$.

3(3). $|A-C|+|B-C| = |(A+B)-C|+|A-B|C$.

3(4). $|A-C|+|B-C| = |AB-C|+|A-B|\overline{C}$.

4(1). $|A-B| < A+B$.

4(2). $|A-C| < |A-B|+|B-C|$.

4(3). $|A-C||B-C| < |(A+B)-C| < |A-C| + |B-C|$.

4(4). $|A-C||B-C| < |AB-C| < |A-C|+|B-C|$.

5. $(A+B+C+\cdots+N)-ABC\cdots N$
$$= |A-B|+|A-C|+\cdots+|A-N|.$$

The standard type of equation in symbolic logic having a unique solution is of the form

$$X\overline{A}+\overline{X}A = B.$$

Using our notation, this becomes $|X-A| = B$ or by 2(1) cor. the solution is $X = |B-A| = B\overline{A}+\overline{B}A$.

The properties under 4 are modular properties which are immediate deductions from the equations under 1(1), 2(2) and 3. Property 5 will be useful to us in what follows.

Definition.—A sequence of classes $S_1, S_2, \cdots, S_n, \cdots$ is said to be a standard sequence, or S-sequence, if $S_n > S_{n+1}$ for all values of n and if $\lim S_n = 0$, that is, if each class includes the next and if there is no element common to all.

A sequence is said to be decreasing if each class includes the next and to be increasing if each class is included in the next.

In this paper inclusion denotes logical inclusion, that is to say, equality is also inclusion in the sense that if $A = B$, then $A > B$ and $B > A$.

Limit.—A sequence of classes $A_1, A_2, \cdots, A_n, \cdots$ is said

to possess a limiting class A if an A can be found such that $|A - A_n| < S_n$ for all values of n, where S_n is the nth member of a standard sequence. We proceed to prove that this definition coincides in content with that given by Borel (see reference above).

Firstly, if the sequence is decreasing and A is its limit by Borel,

$$|A - A_n| = A_n - A < A_n - A$$

and $A_n - A$ forms an S-sequence. Secondly, if the sequence is increasing and A is its limit by Borel,

$$|A - A_n| = A - A_n < A - A_n$$

and $A - A_n$ forms an S-sequence. Thirdly, if in general the sequence possesses a Borel limit A,

$$\begin{aligned}
&|(A_m + A_{m+1} + \cdots + A_{m+p}) - A_n| \\
&\quad < |A_n - A_m| + |A_n - A_{m+1}| + \cdots + |A_n - A_{m+p}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{Prop. } 4(3)] \\
&\quad < |A_n - A_{n+1}| + |A_n - A_{n+2}| + \cdots + |A_n - A_{m+p}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } m > n \\
&\quad < (A_n + A_{n+1} + \cdots + A_{m+p}) - A_nA_{n+1}\cdots A_{m+p} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{Prop. } 5] \\
&\quad < (A_n + A_{n+1} + \cdots) - A_nA_{n+1}\cdots, \\
&|(A_m + A_{m+1} + \cdots) - A_n| \\
&\qquad < (A_n + A_{n+1} + \cdots) - A_nA_{n+1}\cdots \quad \text{if } m > n.
\end{aligned}$$

But the right-hand side forms a decreasing sequence and has the limit 0 if the limit of sequence A_n exists according to Borel. Thus the right-hand side is an S_n of an S-sequence. $|(A_m + A_{m+1} + \cdots) - A_n| < S_n$ for all values of $m > n$. Hence in the limit as m increases indefinitely

$$|\text{Borel limit } A - A_n| < S_n.$$

Hence the sequence A_n has the Borel limit A, according to our definition, whenever it has a limit by Borel's definition. Again, suppose that it has a limit A by our definition

$$|A - A_n| < S_n$$

of some S sequence.

$$
\begin{aligned}
&|(A_n + A_{n+1} + \cdots + A_{n+p}) - A| \\
&\quad < |A - A_n| + |A - A_{n+1}| + \cdots + |A - A_{n+p}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{Prop. } 4(3)] \\
&\quad < S_n + S_{n+1} + \cdots + S_{n+p} \\
&\quad < S_n.
\end{aligned}
$$

Taking the limit as p increases indefinitely, if

$$B_n = A_n + A_{n+1} + \cdots$$
$$|B_n - A| < S_n.$$

But B_n forms a decreasing sequence and has a limit B which is the upper limiting set of the sequence A_n. Hence

$$|B - B_n| < S_n'$$

of some S-sequence; in fact

$$
\begin{aligned}
S_n' &= B_n - B \\
|B - A| &< |B - B_n| + |B_n - A| \quad [\text{Prop. } 4(2)] \\
&< S_n' + S_n
\end{aligned}
$$

for all values of n. But $|B - A|$ is independent of n, hence

$$
\begin{aligned}
|B - A| &= 0 \\
B &= A \qquad\qquad [\text{Prop. } 1((3].
\end{aligned}
$$

Thus the upper limiting set $= A$. Similarly it can be proved that the lower limiting set $= A$ and therefore the sequence A_n has a limit according to Borel and this limit is A, the same as the limit by our definition.

Property 6(1).—The necessary and sufficient condition that a sequence of sets or classes A_n has a limit A according to Borel is that $|A - A_n| < S_n$ of some standard sequence.

Property 6(2).—The necessary and sufficient condition that a sequence of sets or classes A_n has a limit is that

$$|A_n - A_{n+p}| < S_n$$

of some S-sequence, the same for all values of p. For if the sequence A_n possesses a limit A

$$
\begin{aligned}
&|A - A_n| < S_n, \quad |A - A_{n+p}| < S_{n+p}, \\
&|A_n - A_{n+p}| < |A - A_n| + |A - A_{n+p}| \\
&\qquad < S_n + S_{n+p} \\
&\qquad < S_n.
\end{aligned}
$$

If, on the other hand,

$$|A_n - A_{n+p}| < S_n$$

$$\begin{aligned}(A_n + A_{n+1} + \cdots + A_{n+p}) - A_nA_{n+1}\cdots A_{n+p} \\ = |A_n - A_{n+1}| + |A_n - A_{n+2}| + \cdots + |A_n - A_{n+p}| \\ < S_n + S_n + \cdots + S_n \\ < S_n.\end{aligned}$$

In the limit when p increases indefinitely

$$(A_n + A_{n+1} + \cdots) - A_nA_{n+1}\cdots < S_n.$$

The left member is a sequence decreasing with n and it must have the limit 0 for S_n has the limit 0. Consequently the sequence A_n has a limit according to Borel.

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ON THE PRINCIPAL UNITS OF AN ALGEBRAIC DOMAIN $k(\mathfrak{p}, \alpha)$.

BY DR. G. E. WAHLIN.

(Read before the American Mathematical Society, April 10, 1914.)

Introduction.

THE following paper is the result of an investigation of a problem connected with the representation of the algebraic numbers in the form $\pi^\alpha\omega^\beta e^\gamma$.*

Throughout the discussion I shall use the following notation. By p I mean a rational prime and by $\mathfrak{p}$ any prime divisor of p. f is the degree of $\mathfrak{p}$, i. e., $N(\mathfrak{p}) = p^f$ and $\mathfrak{p}^\sigma$ is the highest power of $\mathfrak{p}$ contained in p. By π I mean a prime number of the domain $k(\mathfrak{p}, \alpha)$, where α is an arbitrary algebraic number. The numbers of $k(\mathfrak{p}, \alpha)$ are then of the form $a_\rho\pi^\rho + a_{\rho+1}\pi^{\rho+1} + \cdots$. A number in which $\rho = 0$ and a_ρ is relatively prime to $\mathfrak{p}$ is called a unit and in particular if $a_\rho = 1$ it is called a principal unit.

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* Hensel, *Crelle's Journal*, vol. 145.

The Equation $x^{p^n} - E = 0(\mathfrak{p})$.

For the present we shall let E be any unit of $k(\mathfrak{p}, \alpha)$. From the general theory of algebraic numbers* we know that there exists a certain rational integer μ such that the equation

$$x^{p^n} - E = 0(\mathfrak{p}) \tag{1}$$

has a solution in $k(\mathfrak{p}, \alpha)$ if the congruence

$$x^{p^n} - E \equiv 0 \bmod \mathfrak{p}^{\mu+1} \tag{2}$$

has a solution in this domain. The present section is devoted to the computation of the value of μ.

This determination of μ can be accomplished by making use of a known theorem.†

Since E is a unit, it follows that any solution E_1 of (2) is also relatively prime to $\mathfrak{p}$. Therefore if we put $F(x) = x^{p^n} - E$ and denote its ith derivative by $F^{(i)}(x)$ we see that the order of

$$F^{(i)}(E_1)/i! = \frac{p^n(p^n - 1)\cdots(p^n - i + 1)}{i!} E_1^{p^n - i}$$

is the same as the order of $C^{(i)} = p^n!/i!(p^n - i)!$.

The order of $m!$ in $k(p)$ is $(m - S_m)/(p - 1)$‡ where S_m is the sum of the coefficients in the reduced p-adic representation of m. Hence since $S_{p^n} = 1$ we know that in $k(p)$ the order of $C^{(i)}$ is

$$\frac{p^n - 1}{p - 1} - \frac{i - S_i}{p - 1} - \frac{p^n - i - S_{p^n - i}}{p - 1} = \frac{S_i + S_{p^n - i} - 1}{p - 1}.$$

Let us denote the order of i by ρ and suppose that in its reduced p-adic representation $i = a_\rho p^\rho + a_{\rho+1}p^{\rho+1} + \ldots + a_{n-1}p^{n-1}$. Since $i \leqq p^n$ the representation cannot have a term containing a higher power of p than p^{n-1}, excepting in the case where $i = p^n$ and then the order of $C^{(i)}$ is zero. The number p^n can be written in the form $p \cdot p^\rho + (p - 1)p^{\rho+1} + \ldots + (p - 1)p^{n-1}$, and hence

$$p^n - i = (p - a_\rho)p^\rho + (p - a_{\rho+1} - 1)p^{\rho+1} + \ldots + (p - a_{n-1} - 1)p^{n-1},$$

* Hensel, Theorie der algebraischen Zahlen, Kap. 4, § 4. (The method there used by Professor Hensel can be extended to any domain.)

† Ibid., Kap. 4, § 4, pp. 72–74.

‡ Ibid., p. 111.

which as is easily seen is also in the reduced form. Hence

$$S_i = a_\rho + a_{\rho+1} + \ldots + a_{n-1}$$

and

$$S_{p^n-i} = p - a_\rho + p - a_{\rho+1} - 1 + \ldots + p - a_{n-1} - 1$$

and

$$S_i + S_{p^n-i} = (n - \rho)p - (n - \rho - 1),$$

whence we have

$$(S_i + S_{p^n-i} - 1)/(p - 1) = n - \rho.$$

Since $\mathfrak{p}^\sigma$ is the highest power of $\mathfrak{p}$ in p we see now that $\rho^{(i)}$, the order of $C^{(i)}$ in $k(\mathfrak{p}, \alpha)$, is equal to $\sigma(n - \rho)$.

If we now form the expression $(i\rho' - \rho^{(i)})/(i - 1)$* we see that this is equal to

$$\sigma\frac{ni - (n + \rho)}{i - 1} = \sigma\left(n + \frac{\rho}{i - 1}\right)$$

since $\rho' = n\sigma$. The value of μ sought is the largest integer which is less than or equal to

$$\max \sigma\left(n + \frac{\rho}{i - 1}\right) \text{ for } i = 2, 3, \ldots, p^n.$$

Since n and σ are independent of i, it is evident that this maximum occurs when $\rho/(i - 1)$ is maximum and we shall therefore determine the value of i for which such is the case.

If we first consider the values of i of a given order ρ it is clear that $\rho/(i - 1)$ is maximum when i is minimum and hence when $i = p^\rho$ and the maximum value of $\rho/(p^\rho - 1)$ as ρ varies over the numbers $1, 2, \ldots n$ is therefore the same as the maximum value of $\rho/(i - 1)$ as i varies over the numbers $2, 3, \ldots p^n$. We note here that for $1 < i < p$, $\rho = 0$ and $\rho/(i - 1) = 0$.

Let us now turn our attention to the expression

$$\psi(\rho) = \rho/(p^\rho - 1).$$

Differentiating, we have

$$\psi'(\rho) = \frac{p^\rho - 1 - \rho p^\rho \log p}{(p^\rho - 1)^2} = \frac{p^\rho(1 - \rho \log p) - 1}{(p^\rho - 1)^2}.$$

* Hensel, Theorie der algebraischen Zahlen, Kap. 4, § 4.

If $p > 2$, $\log p > 1$ and hence, since $\rho \geqq 1$, $\psi'(\rho) < 0$. The function $\psi(\rho)$ is therefore a decreasing function for $\rho \geqq 1$ and the maximum value in the required interval therefore occurs when $\rho = 1$. This maximum value is $1/(p-1) > 0$ and since for $1 < i < p$, $\rho/(i-1) = 0$, $1/(p-1)$ is the maximum value of $\rho/(i-1)$. If $p = 2 < e < 4$, since $e^{1/2} < 2$ we have $\frac{1}{2} < \log 2 < 1$ and hence for $\rho \geqq 2$ we have $\rho \log 2 > 1$ and as before $\psi'(\rho) < 0$. Therefore for $\rho \geqq 2$, $\psi(\rho)$ is decreasing and must be maximum, in the given interval, when $\rho = 2$. Hence when ρ takes the values $1, 2, \ldots, n$, $\psi(\rho)$ must be maximum either at $\rho = 1$ or $\rho = 2$.

$$\text{For } \rho = 1, \psi(\rho) = \frac{1}{2-1} = 1.$$

$$\text{For } \rho = 2, \psi(\rho) = \frac{2}{4-1} = \tfrac{2}{3}$$

and hence, as in the preceding case, the maximum value occurs when $\rho = 1$ and again the maximum is $1/(p-1)$. Therefore

$$\max\left(\frac{i\rho' - \rho^{(i)}}{i-1}\right) = \sigma\left\{n + \frac{1}{p-1}\right\}$$

and if we put $k = [\sigma/(p-1)]$ we have

$$\mu = n\sigma + k.$$

A Certain Residue Group in $k(\mathfrak{p}, \alpha)$.

We shall suppose that the domain $k(\mathfrak{p}, \alpha)$ contains all the p^rth roots of unity while no primitive p^{r+1}th root of unity is contained in it. We shall in this discussion need the number μ of the preceding section for the special case when $n = r + 1$ and shall therefore put $\mu = r\sigma + \sigma + k \geqq 1 + k$.

Every principal unit E of our domain is, modulo $p^{\mu+1}$, congruent to one and only one of the $p^{\mu f}$ units $1 + a_1\pi + a_2\pi^2 + \cdots + a_\mu\pi^\mu$ where the a_i vary independently over the p^f numbers of a complete residual system modulo $\mathfrak{p}$. Since the product and quotient of two principal units are principal units it is evident that these residues and hence the E's themselves form an abelian group of order $p^{\mu f}$ with respect to the modulus $p^{\mu+1}$. This group we shall denote by G. Since G is an abelian group we know that it is the product of cyclic groups. These cyclic groups we shall denote by $C_1, C_2, \cdots C_h$, and the order of C_i we shall denote by p^{r_i}. (The order must

be a power of p since it is a divisor of $p^{\mu f}$.) We shall moreover assume that $r_1 \geqq r_2 \geqq r_3 \geqq \cdots \geqq r_h$.

Let m be that one of the numbers $1, 2, \cdots, h$ such that $r_m > r \geqq r_{m+1}$ or if $r_h > r$, $h = m$. We shall first see that r cannot be greater than r_1. G cannot contain an element of period greater than p^{r_1} and hence if R is a primitive p^rth root of unity, it is an element of G and therefore $R^{p^{r_1}} \equiv 1 \bmod p^{\mu+1}$ and hence also modulo p^{k+1}, since $\mu \geqq k+1$. But since $R^{p^{r_1}} \equiv 1 \bmod p^{k+1}$ it is an exponential unit* and we can therefore write $R^{p^{r_1}} = e^{\gamma}(\mathfrak{p})$. By raising both members of this equation to the power p^{r-r_1} we have $e^{\gamma p^{r-r_1}} = 1(\mathfrak{p})$ and hence $\gamma p^{r-r_1} = 0\ (\mathfrak{p})$ and $\gamma = 0\ (\mathfrak{p})$. But then $R^{p^{r_1}} = e^{\gamma} = 1(\mathfrak{p})$ and since R is a primitive p^rth root of unity this is impossible unless $r \leqq r_1$.

In the same way it follows that for $t < r$, $R^{p^t} \not\equiv 1 \bmod \mathfrak{p}^{\mu+1}$ and hence R and its powers form a cyclic subgroup of G, of order p^r.

If $r = r_1$ it is evident, from the proof of the theorem, that every abelian group can be written as the product of cyclic subgroups,† that we can put $C_1 = C$ where C is the cyclic group generated by R. If however $m > 1$ we shall next see that no power of R excepting R^{p^r} is modulo $p^{\mu+1}$ congruent to a number in the product $C_1 \cdot C_2 \cdots C_m$.

Let us denote by E_i any generator of the cyclic group C_i and let us suppose that

$$(3) \qquad E_1^{n_1 p^{\lambda_1}} \cdot E_2^{n_2 p^{\lambda_2}} \cdots E_m^{n_m p^{\lambda_m}} \equiv R^{np^{\lambda}} \bmod \mathfrak{p}^{\mu+1},$$

where we assume that $n, n_1, n_2, \cdots, n_m$ are rational integers relatively prime to p and $0 \leqq \lambda < r$ and $0 \leqq \lambda_i < r_i$ $(i = 1, 2, \cdots m)$. By raising both members of (3) to the power $p^{r-\lambda}$ we have

$$(4) \quad E_1^{n_1 p^{\lambda_1 + r - \lambda}} \cdot E_2^{n_2 p^{\lambda_2 + r - \lambda}} \cdots E_m^{n_m p^{\lambda_m + r - \lambda}} \equiv 1 \bmod \mathfrak{p}^{\mu+1}$$

and from the fact that G is an abelian group and $C_1, C_2, \cdots, C_h$ the base we know that this is possible when and only when the exponent of each E_i is divisible by p^{r_i}. Hence $\lambda_i + r - \lambda \geqq r_i$ and since for $i \leqq m$, $r_i > r$, we have $\lambda_i \geqq r_i - r + \lambda > \lambda$. If we now let $l = \min(\lambda_1, \lambda_2, \cdots \lambda_m)$ and put

$$E = E_1^{n_1 p^{\lambda_1 - l}} \cdot E_2^{n_2 p^{\lambda_2 - l}} \cdots E_m^{n_m p^{\lambda_m - l}}$$

* Hensel, *Crelle's Journal*, vol. 145, pp. 94–95.

† Weber, Algebra, vol. II, pp. 3, 38–45.

we can write (3) in the form

$$E^{p^l} \equiv R^{np^\lambda} \bmod \mathfrak{p}^{\mu+1}.$$

Since $\lambda_2 > \lambda$ it follows that $l > \lambda$.

If we now put $t =$ minimum $(l, r+1)$ and use the result of the first part of this paper we can from the last congruence conclude that the equation

$$x^{p^t} = R^{np^\lambda} (\mathfrak{p}) \tag{5}$$

has a solution in $k(\mathfrak{p}, \alpha)$. Let us denote this solution by $\mathfrak{A}$. Then $\mathfrak{A}^{p^{t+r-\lambda}} = 1(\mathfrak{p})$. Since R is a primitive p^rth root of unity and n is relatively prime to p, R^n is also a primitive p^rth root of unity and hence

$$\mathfrak{A}^{p^{t+r-\lambda-1}} = (R^n)^{p^{r-1}} \neq 1(\mathfrak{p}).$$

$\mathfrak{A}$ is therefore a primitive $p^{t+r-\lambda}$th root of unity which is contained in $k(\mathfrak{p}, \alpha)$.

But we have seen that $l > \lambda$ and have assumed that $\lambda < r$ and hence $r + 1 > \lambda$ and consequently $t = \min(r+1, l) > \lambda$ and $t + r - \lambda > r$. But this contradicts our assumption that $k(\mathfrak{p}, \alpha)$ contains no primitive p^{r+1}th root of unity.

Hence (3) is impossible when $\lambda < r$ and hence no power of R excepting $R^{p^r} = R^0$ or power of R^{p^r} can be congruent, modulo $p^{\mu+1}$ to the left hand member of (3).

From this it now follows that in the construction of the base of G we can put $C_{m+1} = C$ and hence have

$$G = C_1 \cdot C_2 \cdots C_m \cdot C \cdot C_{m+2} \cdots C_h.$$

If we put $G_1 = C_1 \cdot C_2 \cdots C_m \cdot C_{m+2} \cdots C_h$, this is also an abelian group and

$$G = G_1 \cdot C. \tag{6}$$

The result may now be summed up in the following

THEOREM: *If the domain $k(\mathfrak{p}, \alpha)$ contains a primitive p^rth root of unity but no primitive p^{r+1} th root of unity, and if we denote by μ the number $r\sigma + \sigma + k$ where σ is the exponent of the prime divisor $\mathfrak{p}$ in p and $k = [\sigma/(p-1)]$, then the abelian group consisting of the principal units of $k(\mathfrak{p}, \alpha)$ modulo $\mathfrak{p}^{\mu+1}$ is the product of an abelian group G_1 and the cyclic group C whose elements are the p^rth roots of unity.*

UNIVERSITY OF ILLINOIS.

PROJECTIVE GEOMETRY.

Synthetic Projective Geometry. By DERRICK NORMAN LEHMER. Boston, 1917. 16mo. xiii+120 pp.

Socrates is a lobster.
Lobsters are mortal; therefore
Socrates is mortal.

IT was from some such example as this that we first learned how a correct conclusion could be drawn from a faulty minor premise. The book before us is based upon a similar syllogism.

"The subject of projective geometry is . . . destined soon to force its way into the secondary schools;"

That which is taught in secondary schools needs a clear and very simple text-book; therefore

Projective geometry needs a clear and very simple text-book.

With this conclusion we most heartily agree, we wish that we might also agree with the minor premise. Alas, our ears ring with the cries of those who would abolish geometry from our high schools altogether for the mathematically worded reason that it does not "function." We suppose that space will continue to exist, even after people have discovered that there is no need to study its properties, and perhaps the nice little book before us will have an archæological importance long after its subject matter has been proved valueless by the refined tests of laboratory psychology.

It *is* a nice little book, in spite of a bad start and certain other faults which we shall point out in detail. Utterly conventional in type, with the classic methods of the projective geometry of the nineteenth century everywhere in evidence, it is clear, interesting, and readable; the simplest and most elementary book on the subject that we know. The figures are rather small, but the general page impression is pleasing, and the proof-reading seems flawless.

The book, we say, makes a bad start; let us explain. The first chapter deals with one-to-one correspondence, an abstract notion no matter how carefully explained. No later than page 3 we read: "If a one-to-one correspondence has been set up between the objects of one set and the objects of another set, then the inference may usually be drawn that they have the same number of elements."

"Usually be drawn," we snort, "and when, pray, may it not be drawn?" We are answered at the bottom of the page.

Two lines AB and $A'B'$ are drawn of different lengths, and put into one-to-one correspondence, "but," says the author, "it would be absurd to infer from this that there are as many points on AB as on $A'B'$."

The conclusion that one would naturally draw from this astonishing statement is that the author has never heard of the modern theory of assemblages, but such a conclusion would be erroneous, for we read, page 21: "It is perfectly possible to set up a one-to-one correspondence between the points of a line and those of a plane."

We receive a second jolt on page 8, where we are told that the totality of points of a line form an infinitude of the first order, while those of a plane form an infinitude of the second order. What is meant is, of course, that the points of a line may be made to depend continuously on one parameter, while those of a plane depend continuously on two parameters. Since, however, it is assumed that the reader's infant mind is not up to understanding such a statement, he is left to puzzle out as best he may how two infinitudes of different orders can be put into one-to-one correspondence. We learn on page 6 that a projective transformation is a continuous transformation and on page 11 we receive this warning: "It must not be forgotten, however, that we are considering only continuous correspondences." It is a wise restriction; let us turn to page 7. "It is easy to set up a one-to-one correspondence between the points in a plane, and the system of lines cutting across two lines which lie in different planes." Let us remember that it has never been suggested that a one-to-one correspondence could have exceptional elements, and the plane in question is the projective plane which has the connectivity one. The points on two lines, however, may be put into one-to-one correspondence with the points on a ruled hyperboloid, and the statement is that it is easy to set up a continuous one-to-one correspondence between the points of a surface of connectivity one and one of connectivity two. Finally we have the problem, page 13; remember that this is one of the first problems set to the geometrical infants for whom the book is written.

"Is the axiom 'The whole is greater than one of its parts' applicable to infinite assemblages?"

What answer is expected? If the author asks in the mood displayed on page 11, where he says that there is a one-to-one correspondence between the points of a line, and those of a plane, the wise pupil will answer that the axiom has lost all validity. But if the author is in the mood of page 6 where he says that it would be absurd to say that there were the same number of points on two lines of different lengths, then the prudent course would seem to be to reply that the axiom is still doing business at the old stand.

Let us epitomize these ill-natured remarks. We are not suggesting that pupils for whom this book is intended should be taught the theory of assemblages, or the analysis situs. Heaven defend us from any such idea! What we do insist is that if it be bad teaching to tell them things which they do not understand, it is worse teaching to tell them things which are not true. Certainly in their case "It is better not to know so many things, than to know so many things that ain't so."

In the second chapter we get a real start in the proper subject matter of the book. The topics dealt with are the fundamental principles of projection and intersection. Of course Desargues's two triangle theorem is of first importance here; let us see what proof the author gives. He starts ahead in the conventional way, taking the case where the two triangles are in different planes, and this case is done up in good shape. Then the proof comes to an abrupt end* with these words:

"If, now, we consider a plane figure, the points P, Q, and R still lie on a straight line, which proves the theorem."

What the author doubtless means is that if one plane approach the other as a limit, the line P, Q, R will approach a definite limiting position in that plane, so that the proof still holds. What he says, however, is that these three points are collinear, even when the triangles are coplanar. If this be not self-evident, why not prove it, since it is the crux of the situation? But if it be self-evident, why bother with the three-dimensional case in a book on plane geometry?

After this, the book runs ahead smoothly enough for sixteen pages, the next difficulty occurring at that most dangerous point, the fundamental theorem of projective geometry:

"*If two projective one-dimensional forms have more than two self-corresponding elements, they are identical.*"

* P. 16.

This theorem has had a long and stormy career. The present author knows well that it can not be proved by projective synthetic methods without the aid of continuity. He therefore introduces, *ad hoc*, an axiom to the effect that a projective correspondence is a continuous correspondence. Unfortunately, even with the timely assistance of this assumption, the proof is not perfect, for it is assumed* that if at each stage of a certain process the interval which contains a given point is diminished, then the interval itself can be made as small as we please.

The fourth and fifth chapters deal with point rows and linear pencils of the second order, i. e., point and line conics. Pascal and Brianchon play leading rôles, the whole discussion goes ahead smoothly enough. The same may be said of the following chapter which deals with poles and polars, the only exception being a small mistake which is not unknown in elementary text-books of analytic geometry. The polar of a point is defined in a fashion which is inadmissible when the point lies on the conic.† Consequently the following fundamental theorem whereby if one point lie on the polar of a second, the second is on the polar of the first, suffers an exception when one of these points is on the conic. It is better to define the tangent as the polar of a point of the curve; the fundamental theorem then suffers no exception.

The eighth chapter, which might better have been put in the seventh place, deals with the properties of involutions. The author says in the preface that he has never felt satisfied with the usual treatment of involutions by means of circles and anharmonic ratios; the present treatment represents his idea of how the subject may be made easier and more consistent. His treatment is certainly purely projective, and entirely logical, except for the unproved statement‡ that an involution must have two double points or none. It seems to us, however, that the order of topics is artificial, and in consequence, some of the proofs quite needlessly hard. He begins with point involutions; three pairs of collinear points are said to be in involution if they lie on three pairs of opposite sides of a complete quadrangle. Then comes a proof of Desargues's involution theorem for conics which is quite

* P. 32.
† P. 57.
‡ P. 73.

complicated, and lastly what the author calls the fundamental theorem, namely, that if in two projective fundamental one-dimensional forms, a single pair correspond interchangeably, the same is true of every pair. The proof of this covers a whole page, and even then the last details are left to the reader. Then we have dual definitions and theorems for involutions of lines. Finally we deduce from the fundamental theorem that a transversal will meet an involution of concurrent lines in an involution range of points. All this is, as we said, a perfectly legitimate way to handle the subject; we can not think that it is a particularly easy one, for the proofs of these same theorems in such a book as Veblen and Young's "Projective Geometry"* even when written out "in a language understanded of the people" are demonstrably shorter than those of the present author.

Chapters VII and IX deal with metrical properties of conics and of involutions respectively. Purists will incline to look upon the introduction of this material as a blemish on the beauty or consistency of the structure. Our own view is just the opposite. These metrical ideas are introduced in illustration of special cases of the theorems developed, not in the course of the logical structure. The chief reason why we welcome them is, however, a didactic one. There can be no doubt that the young geometer feels more interested and more at home when he is dealing with metrical theorems, than when he is occupied with exclusively projective ones. The former certainly come more nearly within his ordinary range of interests. Moreover, the great danger is that projective geometry may appear as a subject apart, but slightly connected with any other mathematical branch. This danger is somewhat obviated when the method of approach is algebraic, as the student sees the connection between the geometric theorems and their algebraic formulation. When, however, as in the present instance, the treatment is purely synthetic, the subject is likely to remain entirely hanging in the air, unless it be tied to the earth by being linked up with more familiar metrical material. We therefore welcome these two chapters, IX being especially interesting and valuable.

The last chapter gives a brief résumé of the history of projective geometry. The author's idea is that the time to learn the history of a subject is after one has found out what the subject is. There is something to be said for this view.

*Vol. 1, Boston, 1910, especially pp. 102, 146.

Now we have exhausted our store of invective. Let us close as we began by saying that it is a clear and interesting little book, whose appearance we heartily welcome.

J. L. COOLIDGE.

NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. London, Longmans, Green and Company, 1916. 16mo. vii+179 pp.

MORE than a score of years ago a writer of fluent pen published a short article with the title "Is the Non-Euclidean Geometry Inevitable?" His decision was affirmative, his interpretation of the question being "Are the conclusions of non-euclidean geometry inevitable?" We have learnt that they are, but we have also learnt that both the conclusions and the consequences are unavoidable, and that books dealing with the subject must be expected with certainty and met with fortitude. The number already published is already large, as we see by Somerville's compendious bibliography,* and the desire to publish others is so strong that not even the war can choke it. But there is always room at the top, and if a newcomer does not succeed in getting there himself, he raises the existing leader so much the higher. The book before us is a good one. We shall cheerfully damn it in detail later on, let us first praise it in general.

To begin with it is an interesting book. The field covered is not wide. Some things are omitted which we regret to miss, but nothing is included which might better have been left out. The choice of material seems largely guided by the principle that the book was primarily written for teachers, and the author has not shut his eyes to the fact that the average Anglo-Saxon mathematical teacher is too busy to read a long mathematical book, and too weak scientifically to understand a deep one. In consequence of this, certain standard topics like the realization of non-euclidean geometry on surfaces of constant total curvature or the subtleties of the Cayleyan metric are passed over in silence. The most serious omissions come from the restriction to plane geometry. In euclidean

* London, Harrison, 1911.

geometry, the most interesting elementary facts about the geometry of three dimensions may be inferred by the light of nature from what happens in the plane, but in non-euclidean space there are exceptions to this convenient rule. For instance, there is nothing in the geometry of the elliptic plane that foreshadows the properties of Clifford's skew parallel lines in space. These parallels constitute one of the most novel and interesting figures in all of non-euclidean geometry; it is a pity that a reader whose knowledge is bounded by the present book must remain in ignorance of them. It is pleasant to note in this connection that the author has a keen didactic instinct. Whenever he is forced to give a proof that is long or difficult, he divides the work into stages of reasonable length, explaining clearly just what is accomplished in each stage.

The first, yes, and much the greatest difficulty with which the writer on non-euclidean geometry has to deal is that of the fundamental assumptions or axioms. The present writer has met this difficulty in exemplary fashion. He recognizes that it would be equally unwise to give a long analysis of axioms according to modern abstract principles, or merely to say that the axioms are Euclid's except the parallel axiom. What he does in fact amounts to taking Hilbert's system, but as all readers can not be presumed to be familiar with this, he puts the matter somewhat differently. Euclid's axioms are retained except the one about parallels, but they are pieced out by assuming the first congruence theorem for triangles, by Hilbert's axiom that a given line segment may be extended a given amount in either direction, Pasch's axiom that a line in the plane of a triangle which passes between two vertices, and does not pass between a second pair nor go through a vertex, must pass between the third pair, and lastly an elaborate continuity axiom for straight lines which, incredible dictu, is never explicitly used. All this is done in a very few pages and completed by certain new constructions for perpendiculars and bisectors. The remainder of the first two chapters is of a historical nature, and follows conventional lines.

The third chapter, dealing with hyperbolic plane geometry, is the strongest in the book. The author takes a legitimate pride in the fact that he nowhere uses continuity in this chapter, but builds with not a little skill on the remaining

axioms. No construction is ever used whose possibility and correctness has not previously been shown. He makes good use of the formulas which connect the parts of a right triangle with those of a quadrilateral with three right angles, formulas which are fundamental in problems of construction in the hyperbolic plane, but which are given scant notice in many of the text-books. The only adverse comment we feel inclined to make is that in spots the logical structure is so very delicate, that an inattentive reader might suspect the existence of mistakes that are not there. The definition of "equidistant curve" on page 83 is incorrect, as it gives only half of such a curve. And thus we reach page 91 and the end of Chapter III and the middle of the book. It would have been a far more flawless piece of work if the author had written "Finis" at that point.

The fourth chapter deals with hyperbolic trigonometry. The author, contrary to the custom of certain other elementary writers, develops this by two-dimensional methods only, the fairy godmother that smoothes out all obstacles being the limiting curve or orocycle. This has been defined correctly on page 80 and on the following page we are told accurately what is meant by congruent orocyclic arcs. From this definition we may safely infer the meaning of a rational ratio of two such arcs "Right and jest, jest and right," as the immortal Disko Troop remarked. But we know nothing of incommensurable arcs on orocycles, nor do we yet know that an orocycle is a continuous curve, and when we read on page 93 "if the arcs are incommensurable we reach the same conclusion by proceeding to the limit," we feel as if we had been rudely awakened by the whole logical structure crashing down about our ears. The same seductive process of proceeding to the limit is used again on pages 128 and 139. It is so fatally easy! But why give at the outset an elaborate continuity axiom and why employ the logical rigor of the strictest sect of the pharisees during the whole first part of the book, if one is coming to this at the last? A similar lapse occurs in the handling of the equation*

$$\tanh a = \cos f(\alpha).$$

It is shown that f decreases as a increases, and it is assumed without further ado that it is continuous and differentiable!

* Pp. 106ff.

Chapter V deals with the differential of length in the hyperbolic plane, and measures of area. An area is taken as a primary concept and not further defined.

Chapter VI brings us at last to the elliptic geometry. It is brief and follows conventional methods. There is a slight slip on page 132, where the author makes use of the theorem that an exterior angle of a triangle is greater than either opposite interior angle. This is only true in the elliptic plane if the region be sufficiently small. On page 133, line 3, *FB* should read *FP*. This is the only printer's error which we have noticed in the book.

Chapter VII deals with the elliptic plane trigonometry, and is the most difficult chapter in the work. The method followed is that originally devised by Gérard, although in spots, as on page 140, the present author omits some tedious if essential details. The whole treatment suggests a didactic question of not a little interest to all who undertake to teach non-euclidean geometry. In developing the elementary parts of the subject, one may follow one of two different methods. The first is to develop a general geometry as far as possible, and to give the theorems characteristic of the particular geometries only after the general theorems have all been put in evidence. Similarly in trigonometry, a general set of formulas is derived suitable to all three classical geometries, and the distinction of one from the other depends on the value of the space constant. The second method consists in making full use in the case of each geometry of the features characteristic of that geometry. Limiting ourselves to the consideration of recent text-books we may say that Killing* and the reviewer† have followed the first method, while not only the present author, but Liebmann,‡ Manning,§ and Sommerville|| have followed the second. It is largely a question of ideal. The first method lays emphasis upon the points of similarity of the three geometries, the second emphasizes their points of distinction. The first method is shorter, as the fundamental equations have to be deduced but once, the second approximates more closely to an ideal which the late Gaston Darboux once explained to the reviewer in about these words:

* Grundlagen der Geometrie, Paderborn, 1893, especially pp. 80ff.
† The Elements of Non-Euclidean Geometry, Oxford, 1909.
‡ Nicht-euklidische Geometrie, Second Ed., Leipzig, 1912.
§ Non-Euclidean Geometry, Second Ed., Boston, 1915.
|| The Elements of Non-Euclidean Geometry, London, 1914.

"It is the distinctive characteristic of geometry, not to have a general method, but to find in the problem itself the methods best suited to its solution."

Let us note in passing an unintentional pleasantry on page 143 where we read "SOs is acute."

Our book closes with a peculiarly interesting chapter, No. 8. Most writers on non-euclidean geometry feel the necessity of showing that the subject will really "work" by exhibiting examples of one or more geometrical systems which obey just the desired hypotheses. This is usually done by a discussion of the geometry on certain surfaces of constant curvature, but this procedure is rather blind to a reader who does not know differential geometry. The present writer departs from this precedent, and gives a bird's-eye view of the three geometries at once by building the geometry of what is called "nominal lines" and "nominal points." A nominal line is nothing more nor less than a euclidean circle with regard to which a chosen fixed point has a preassigned power, or, in the limiting case, a line through that point. The credit for that idea is given to Poincaré, "Science et Hypothèse," and the reviewer, for one, had always supposed until recently that the idea was entirely derived from the lamented French geometer. As matter of fact a good share of the credit should be given to an earlier and less-known writer, DePaolis, who exhibited in 1878 a one-to-two conformal transformation from the non-euclidean to the euclidean plane, where non-euclidean lines passed over into euclidean circles orthogonal to a fixed circle, the square of whose radius was positive, negative, or zero.* The difference between this and the Poincaré scheme is largely a question of phraseology. Let us explain briefly how the plan works.

We take a fixed point O and a fixed number k, positive, negative, or zero. We define as a nominal line a line through O or a circle with regard to which O has the power k. If k be positive, we take a fixed circle with center O and radius $\sqrt{k}$; the nominal lines all cut it orthogonally, and two points determine a nominal line, unless they be inverse in the fixed circle. All points within, or on, this circle are defined as nominal points, and two nominal points will always determine just one nominal line; we have an excellent example of the

* "La trasformazione piana doppia, etc.," *Memorie della R. Accademia dei Lincei*, series 3, vol. 2 (1878).

hyperbolic plane. If k be negative, the nominal lines are circles which meet a fixed circle with center 0 and radius $\sqrt{-k}$ in diametrically opposite points, or, in the limiting case, lines through O. Two points will determine a nominal line, unless one be the reflection in O of the inverse of the other in this circle. We take as nominal points the points within the circle, and the pairs of diametrically opposite points on the circle; we have an admirable elliptic plane. Lastly, in the case where k is zero we take as nominal lines the lines or circles through O, and as nominal points the totality of finite points except O. This gives a good example of the euclidean plane. With regard to distances, we may follow the author and define them Cayley fashion by the logarithms of certain cross ratios, but this involves rather higher mathematical considerations that have been introduced before, and breaks down entirely for the euclidean case. The author's treatment of this last case is simply lamentable. He says:*

We define the nominal length of a nominal line as the length of the rectilinear segment to which it corresponds . . . the nominal length of a nominal segment is unaltered by inversion with regard to a circle of the system.

What this definition means, we do not know, as there is no indication of just how a nominal line *corresponds* to a rectilinear segment. The conclusion would seem to be that the nominal length of a nominal line is defined as the length of either the arc or the chord of the circle which is that nominal line. But a moment's thought shows that neither of these is invariant under the inversions in question. This erroneous statement suggests, however, another way of putting the thing which covers all three cases.

Let two nominal segments be defined as "congruent" if they may be transformed into one another by a succession of inversions or reflections in nominal lines. It is easy to see that no segment is congruent to a part of itself. If a segment be split in two, we may define the process of addition by saying that the length of the sum shall be the sum of their lengths. It can then be shown how any length can be measured in terms of any other.

Let us point out, in conclusion, that the whole scheme of nominal lines, beautiful as it is, represents a change on the

* P. 158.

author's part from the second to the first of those ideals in non-euclidean geometry which we dwelt on above.

J. L. COOLIDGE.

CORRECTION.

SPEAKING of M. Nau's translation of Sebokht's reference to the Hindu-Arabic numerals, in the BULLETIN for May, 1917 (volume 23, page 366), I remarked that no report of the matter "seems as yet to have appeared in English." My attention has since been called to the fact that Professor L. C. Karpinski announced the discovery in *Science* for June 21, 1912. While this does not concern Mr. Ginsburg's valuable note on the work and influence of Sebokht, the correction as to the publication of the extract in English should be made.

DAVID EUGENE SMITH.

NOTES.

THE June number (volume 18, number 4) of the *Annals of Mathematics* contains the following papers: "Fermat's last theorem and the origin and nature of the theory of algebraic numbers," by L. E. DICKSON; "The modified remainders obtained in finding the highest common factor of two polynomials," by A. J. PELL and R. L. GORDON; "Nomograms of adjustment," by L. I. HEWES; "Closed algebraic correspondences," by A. A. BENNETT; "The intersections of a straight line and hyperquadric," by J. L. COOLIDGE; "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivatives," by W. B. FITE; "Conjugate planar nets with equal invariants," by L. P. EISENHART.

AT the meeting of the Edinburgh mathematical society on May 11 the following papers were read: By L. R. FORD: "A geometrical proof of a theorem by Hurwitz and Borel"; by D. G. TAYLOR: "Geometrical illustrations of cyclant substitutions."

At the meeting of the London mathematical society held on May 3 the following papers were read: By G. H. Hardy, "Sir George Stokes and the theory of uniform convergence"; by W. P. Milne, "A symmetrical condition for ω-apolar triads on a cubic curve."

University of Paris.—The following courses in mathematics are being given during the present semester (March 1–July 15).—By Professor L. C. Guichard: General principles of analytic geometry with applications to the theory of cyclides and correlative surfaces, two hours.—By Professor E. Picard: Theory of integral equations with applications to problems in mathematical physics, two hours.—By Professor E. Goursat: Differential and integral calculus, two hours.—By Professor P. Painlevé: Rational mechanics, two hours.—By Professor L. Vessiot: Formal resolution of the problem of three bodies by means of trigonometric series, and periodic solutions, two hours.—By Professor C. J. de la Vallée-Poussin: Conferences in higher analysis, two hours.

Professor E. Picard was unanimously elected permanent secretary of the Paris Academy of sciences on April 2, as successor to the late Professor G. Darboux. Professor Picard became a member of the academy in 1889, succeeding Halphen.

Professor A. Speiser has been appointed professor of mathematics in the University of Zurich.

At the Massachusetts Institute of Technology, Dr. Joseph Lipka has been promoted to an assistant professorship of mathematics.

At the University of Nebraska, Mr. A. Babbett and Miss Lulu Runge have been promoted to assistant professorships of mathematics.

At Teachers College, Columbia University, assistant professor C. B. Upton has been promoted to an associate professorship of mathematics.

Dr. L. S. Dederick, of Princeton University, has been appointed instructor in mathematics in the U. S. Naval Academy.

BOOK CATALOGUES:—Longmans, Green, and Company, Fourth Avenue and 30th Street, New York, text-books and reference works for colleges and technical schools, 16 pp.—The Macmillan Company, 66 Fifth Avenue, New York, books on mathematics and astronomy, 68 pp.—Galloway and Porter, Cambridge, England, short lists of mathematical and physical books, 77 entries.—Librairie Paul Ritti, 76 Avenue du Maine, Paris, Bulletin périodique, mathématique, 41 entries.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BLICHFELDT (H. F.). Finite collineation groups. Chicago, University of Chicago Press, 1917. 12mo. 12+194 pp. $1.50

DARBOUX (G.). Principes de géométrie analytique. Paris, Gauthier-Villars, 1917. 8vo. 6+520 pp. Fr. 20 00

FERNÁNDEZ BAÑOS (O.). Estudio sintético de los espacios complejos de n dimensiones. (Junta para ampliciòn de estudios e investigaciones cientificas, Laboratorio y seminario matemático, Tomo 2, Memoria 1.) Madrid, 1917. 8vo. 80 pp.

HUDSON (R. G.) and LIPKA (J.) A manual of mathematics. New York, Wiley, 1917. 8vo. 2+132 pp. $1.00

——. A table of integrals. New York, Wiley, 1917. 8vo. 25 pp. Paper. $0.15

LIPKA (J.). See HUDSON (R. G.).

MICHEL (C.). Cours d'algèbre et d'analyse, à l'usage de la classe de mathématiques spéciales et des candidats aux écoles du gouvernement. Paris, Alcan, 1917. 8vo. Fr. 18.00

PRINGSHEIM (A.). Vorlesungen über Zahlen- und Funktionenlehre. 1ter Band, 2te Abteilung: Unendliche Reihen mit reellen Gliedern. Leipzig, Teubner, 1916. 8vo. 222 pp. M. 10.80

II. ELEMENTARY MATHEMATICS.

AUSTIN (F. E.). Preliminary mathematics. Hanover, N. H., F. E. Austin, 1917. 169 pp. $1 20

BELL (A. H.). Algebra, theoretical and applied: a class-book for secondary, higher elementary and technical schools. London, Blackie, 1916. 8vo. 354 pp. 4s. 6d.

CONCINA (U.) e NEPPI-MODONA (A.). Trigonometria piana ad uso dei licei. Con 50 figure e 809 esercizi e problemi. Torino, Casa G. B. Petrini di Giovanni Galizio, 1917.

——. Trigonometria piana e sferica ad uso degli istituti tecnici Con 75 figure e 1000 esercizi e problemi. Torino, Casa G. B. Petrini di Giovanni Galizio, 1917.

COULOM (F.) et WEILL (—.). Nouveau cours de géométrie théorique et pratique. 3e année. Paris, Delagrave, 1917. 12mo. Fr. 2.25

CRONFELT (S.). Lærebog i Aritmetik og Algebra (med et Tillæg om Effekter og Veksler) for Realklassen. København, Steen Hasselbalch (Jacob Erslev), 1916. 8vo. 104 pp. Kr. 1.75

FRATTINI (—.). Lezioni de algebra, geometria e trigonometria piana e sferica con molti esempi sull'intero programma del secondo bienno degl'istituti tecnici (sezione fisico-matematica). Volume 2 per la quarta classe. 3a edizione. Torino, Paravia, 1917.

GONGGRIJP (B.). Logarithmische en goniometrische tafels en bijtafels. Groningen, Noordhoff, 1916. 8vo. 230 pp. Fl. 1.70

HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). First course in algebra. Revised edition. Boston, Ginn, 1917. 10+301 pp. $1.00

LEMOINE (A.). Arithmétique. Cours supérieur et complémentaire. Brevet élémentaire. Livre du maître. Paris, Hachette, 1917. 12 mo. Fr. 4.00

LUBY (W. A.), See HAWKES (H. E.).

MARTINI-ZUCCAGNI (A.). Guida pratica per la risoluzione delle equazioni di 1° e 2° grado con 214 esercizi risoluti. 5a edizione. Livorno, Giusti, 1917. L. 1.50

NEPPI-MODONA (A.). See CONCINA (U.).

SMITH (O. A.). Praktische Regneopgaver for Realklassen. København, Gjellerup, 1916. 24 pp.

TOUTON (F. C.). See HAWKES (H. E.).

WEILL (—.). See COULOM (F.).

WIJDENES (P.). Logarithmen en rententafels. Groningen, Noordhoff, 1916. 8vo. 64 pp. Fl. 0.50

III. APPLIED MATHEMATICS.

BOTTASSO (M.). Analyse vectorielle générale. Volume 4: Astatique. Avec une préface de R. Marcolongo. Pavia, Mattei, 1915. 8vo. 16+159 pp. Fr. 6.00

BRUNT (D.). The combination of observations. Cambridge, University Press, 1917. 8vo. 10+219 pp. 8s.

BURTON (E. F.). The physical properties of colloidal solutions. New York, Longmans, 1916. 7+200 pp. $1.80

CREAGER (W. P.). Engineering for masonry dams. New York, Wiley, 1917. 250 pp. $2.50

DALE (R. B.). Drawing for builders. New York, Wiley, 1916. 200 pp. $1.50

DOYÈRE (C.). Cours pratique de construction navale. 1re partie: Géométrie du navire. Calculs de déplacement et de stabilité. 2e édition, revue et augmentée. Paris, Challamel, 1917. 8vo. Fr. 10.00

DUCHÊNE (E. A.). Flight without formulae: simple discussions on the mechanics of the aeroplane. Translated from the French by J. H. Ledeboer. 2d edition. New York, Longmans, 1917. 8+211 pp. $2.50

GROSSMANN (M.). Elemente der darstellenden Geometrie. (Teubners Leitfaden für den mathematischen und technischen Hochschulunterricht.) Leipzig, Teubner, 1917. 84 pp. M. 2.00

HENDERSON (W. D.). Problems in physics. New York, McGraw-Hill, 1916. 8+205 pp. $1.50

HUDSON (R. G.), LIPKA (J.), LUTHER (H. B.) and PEABODY (D.). The engineers' manual. New York, Wiley, 1917. 310 pp. $2.00

ISENTHAL (A.). See LILIENTHAL (O.).

JAHNKE (E.). See MEHMKE (R.).

JUDGE (A. W.). The properties of aerofoils and aerodynamic bodies. London, Whittaker, 1917. 8vo. 10+298 pp. 15s.

LEDEBOER (J. H.). See DUCHÊNE (E. A.).

LILIENTHAL (G.). See LILIENTHAL (O.).

LILIENTHAL (O.). Bird flight as the basis of aviation: a contribution toward a system of aviation. With a biographical introduction and addendum by G. Lilienthal. Translated from the second edition by A. Isenthal. New York, Longmans, 1917. 8vo. 14+142 pp. $2.50

LIPKA (J.). See HUDSON (R. G.).

LUTHER (H. B.). See HUDSON (R. G.).

MARCOLONGO (R.). See BOTTASSO (M.).

MEHMKE (R.). Leitfaden zum graphischen Rechnen. (Sammlung mathematisch-physikalischer Lehrbücher, herausgegeben von E. Jahnke.) Leipzig, Teubner, 1917. 8+152 pp. M. 4.80

MELAN (J.) and STEINMAN (D. B.). Plain and reinforced concrete arches. 2d edition. New York, Wiley, 1917. 171 pp. $2.00

NOTTAGE (W. H.). The calculation and measurement of inductance and capacity. London, The Wireless Press, 1917. 8vo. 137 pp. 2s. 6d.

PEABODY (D.). See HUDSON (R. G.).

RICCARDO-BRAUZZI (—.). Cours d'aéronautique générale. 1re partie: Etude des questions qui intéressent tous les modes de locomotion aérienne. Paris, Dunod et Pinat, 1917. 8vo. 344 pp. Fr. 30.00

RICHARDSON (O. W.). The emission of electricity from hot bodies. New York, Longmans, 1916. 6+304 pp. $2.75

RUGGLES (C. L.). Stresses in wire-wrapped guns and in gun carriages. New York, Wiley, 1916. 270 pp. $3.00

SERVICE des grands forces hydrauliques (Région des Alpes). Tome VII: Compte rendu des résultats des études et travaux au 31. décembre 1915. Paris, Dunod et Pinat, 1917. 8vo. 481 pp. Fr. 30.00

STEINMAN (D. B.). See MELAN (J.).

WATERBURY (L. A.). Stresses in structural steel angles. New York, Wiley, 1917. 82 pp. $1.25

TWENTY-SIXTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALEXANDER, J. W., II. On the factorization of Cremona plane transformations. Read Dec. 28, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 3, pp. 295–300; July, 1916.

BATEMAN, H. On systems of partial differential equations and the transformation of spherical harmonics. Read Feb. 26, 1916. *Proceedings of the Royal Society of Edinburgh*, vol. 36, Nos. 3–4, pp. 300–312; March, 1917.

BEAL, F. W. A congruence of circles. Read April 24, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 4, pp. 180–187; June, 1916.

BEATTY, S. The inversion of an analytic function. Read April 28, 1917. *Bulletin of the American Mathematical Society*, vol. 23, No. 8, pp. 347–353; May, 1917.

BENNETT, A. A. A case of iteration in several variables. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 17, No. 4, pp. 188–196; June, 1916.

—— An existence theorem for the solution of a type of real mixed difference equation. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 18, No. 1, pp. 24–30; Sept., 1916.

—— Newton's method in general analysis. Read (Southwestern Section) Dec. 2, 1916. *Proceedings of the National Academy of Sciences*, vol. 2, No. 10, pp. 592–598; Oct., 1916.

—— Closed algebraic correspondences. Read (Chicago) Dec. 22, 1916. *Annals of Mathematics*, ser. 2, vol. 18, No. 4, pp. 200–208; June, 1917.

BIRKHOFF, G. D. Infinite products of analytic matrices. Read Dec. 27, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 3, pp. 386–404; July, 1916.

—— Dynamical systems with two degrees of freedom. Read Dec. 27, 1915 and Sept. 5, 1916. *Proceedings of the National Academy of Sciences*, vol. 3, No. 4, pp. 314–316; April, 1917. *Transactions of the American Mathematical Society*, vol. 18, No. 2, pp. 199–300; April, 1917.

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